On a class of Polish-like spaces

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Joint work with Luca Motto Ros

The starting point

From classical to generalized descriptive set theory:

DST:		**GDST:**
Cantor space ${}^{\omega}2$	$\rightsquigarrow$	κ-Cantor space ${}^{\kappa}2$
Baire space ${}^{\omega}\omega$	$\rightsquigarrow$	κ-Baire space ${}^{\kappa}\kappa$
Polish spaces	$\rightsquigarrow$	κ-Polish spaces?

Context: cardinals κ satisfying $\kappa^{<\kappa} = \kappa$.

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If κ regular, $\kappa^{<\kappa} = \kappa$ is equivalent to $2^{<\kappa} = \kappa$, but the latter allows to extend the definition to singular cardinals.

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V. Dimonte, L. Motto Ros and X. Shi, forthcoming paper on GDST on singular cardinals of countable cofinality.

Motivations and goals

Aim: study GDST on λ singular of uncountable cofinality.

What we want:

A suitable class λ-*DST* of Polish-like spaces of weight λ that:

1. includes ${}^{\lambda}2$ and ${}^{\mathrm{cf}(\lambda)}\lambda$.
2. can support most of DST tools and results.
3. for $\lambda = \omega$ gives exactly Polish spaces.
4. goes well with different definitions of λ-Polish for other known cases.

Context: T_3 (regular and Hausdorf) topological spaces, cardinals λ satisfying $2^{<\lambda} = \lambda$.

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Definition

Let λ be a (singular) cardinal of countable cofinality.
A λ-Polish space is a completely metrizable space of weight λ.

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Remark

The λ-Cantor and λ-Baire spaces are metrizable if and only if $\mathrm{cf}(\lambda) = \omega$.

What is known: λ regular

Theorem

Let X be a second countable (T_1, regular) space. Then

- *X is metrizable.*
- *X is Polish if and only if X is strong Choquet.*

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Definition

The *strong Choquet game* on X is played in the following way:

I	V_0, x_0		V_1, x_1		...
II		U_0		U_1	...

- V_α and U_α are nonempty (if possible) open sets.
- $V_\alpha \subseteq U_\beta \subseteq V_\gamma$ for every $\gamma \leq \beta < \alpha < \omega$.
- $x_\alpha \in V_\alpha$ and $x_\alpha \in U_\alpha$ for every $\alpha < \omega$.

The first player **I** wins if $\bigcap_{\alpha<\omega} U_\alpha = \emptyset$, otherwise **II** wins.

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The *strong δ-Choquet game* on X is played in the following way:

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II		U_0		U_1 $\dots$		U_γ	$\dots$

- V_α and U_α are nonempty (if possible) relatively open sets.
- $V_\alpha \subseteq U_\beta \subseteq V_\gamma$ for every $\gamma \leq \beta < \alpha < \delta$.
- $x_\alpha \in V_\alpha$ and $x_\alpha \in U_\alpha$ for every $\alpha < \delta$.

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Let λ be a singular cardinal. There are strong λ-Choquet topological spaces of weight λ with "patological" behaviour.

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Restoring metrizability

Polish		λ-**DST**
Second countablity	$\rightsquigarrow$	*weight* λ
Completeness	$\rightsquigarrow$	*strong* $\mathrm{cf}(\lambda)$-*Choquet*
Metrizability	$\rightsquigarrow$	*?*

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Theorem (Nagata-Smirnov metrization theorem)

Let X be a topological space. Then X is metrizable if and only X admits a σ-locally finite base.

Definition

Let X be a topological space, and $\mathcal{A}$ a family of subsets of X.
We say $\mathcal{A}$ is locally finite if every point $x \in X$ has a neighborhood U intersecting finitely many pieces of $\mathcal{A}$.
We say $\mathcal{A}$ is σ-locally finite if it has a cover $\mathcal{A} = \bigcup_{i \in \omega} \mathcal{A}_i$ of countable size such that each $\mathcal{A}_i$ is locally finite.

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Every base of size λ is λ-Nagata-Smirnov: it can be covered by λ many singletons.

Proposition

Let λ be a cardinal.

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- *If $\lambda = \omega$, λ-DST means Polish.*
- *If λ uncountable of countable cofinality, λ-DST means completely metrizable of weight λ. (proof to be checked)*

Examples and non-examples

Examples of λ-DST spaces:

1. The λ-Cantor and λ-Baire spaces.
2. Completely metrizable spaces of weight λ.
3. For every tree T of density λ and uniform height, $[T]$ with the bounded topology is λ-DST.
4. If X is λ-DST, then $\mathcal{K}(X)$ with the Vietoris topology is λ-DST.
5. Disjoint unions of λ-many λ-DST spaces are λ-DST.
6. Products of $\mathrm{cf}(\lambda)$-many λ_i-DST spaces are $\sup(\lambda_i)$-DST.
7. Open subspaces of a λ-DST are λ-DST.

Non-examples:

1. Products of $> \mathrm{cf}(\lambda)$ many non-trivial spaces are never λ-DST.
2. If $\mathrm{cf}(\lambda) > \omega$, there is a closed subspace of ${}^{\lambda}2$ which is not λ-DST.
3. If $\mathrm{cf}(\lambda) > \omega$, there is a λ-DST space whose perfect part is not λ-DST.

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Theorem ([2, Theorem 7.9])

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Theorem (A., Motto Ros)

Let X be a λ-DST space. There is a continuous surjective function $f : {}^{\mathrm{cf}(\lambda)}\lambda \to X$ and a closed $C \subseteq {}^{\mathrm{cf}(\lambda)}\lambda$ such that $f \restriction C$ is bijective.

We can get more:

Theorem (A., Motto Ros)

Suppose X is a $\mathrm{cf}(\lambda)$-additive λ-DST space and $\mathrm{cf}(\lambda) > \omega$. Then X is homeomorphic to a (super)closed subspace of ${}^{\mathrm{cf}(\lambda)}\lambda$.

(needs some cardinal assumption if λ-singular)

Recall: X is γ additive if the intersection of $< \gamma$ open sets is open.
Recall: C superclosed if $C = [T]$ for T homogeneous in height.
Recall: A tree T is homogeneous in height if every branch has same height.

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V. Dimonte, L. Motto Ros, X. Shi: let X be λ-perfect λ-Polish space.
There is an embedding of ${}^{\lambda}2$ into X with closed image.

Definition: X λ-perfect if no intersetion of $< \mathrm{cf}(\lambda)$ opens has size $< \lambda$.

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Theorem (A., Motto Ros)

Let X be a λ-perfect λ-DST space. There is a continuous injective function from ${}^{\lambda}2$ into X with λ-Borel inverse.

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Super λ-Choquet game: same game as before, but players can play only big open sets (of size $> \lambda$).
Super λ-DST: super λ-Choquet game instead of strong λ-Choquet.

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Theorem (A., Motto Ros)

Let X be a λ-DST space. Then the perfect kernel of X is λ-DST if and only if X is super λ-DST.

Corollary

Let X be super λ-DST. Then $|X| \leq \lambda$ or there is a continuous injective function from ${}^{\lambda}2$ into X.

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