On a class of Polish-like spaces

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Joint work with Luca Motto Ros
The starting point

From classical to generalized descriptive set theory:

**DST:**
- Cantor space $\omega^2 \sim \kappa$-Cantor space $\kappa^2$
- Baire space $\omega^\omega \sim \kappa$-Baire space $\kappa^\kappa$
- Polish spaces $\sim \kappa$-Polish spaces?

**GDST:**

**Context:** cardinals $\kappa$ satisfying $\kappa^{<\kappa} = \kappa$. 
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**GDST:**

Context: cardinals $\kappa$ satisfying $\kappa^{<\kappa} = \kappa$.

Is the assumption $\kappa^{<\kappa} = \kappa$ necessary?

If $\kappa$ regular, $\kappa^{<\kappa} = \kappa$ is equivalent to $2^{<\kappa} = \kappa$, but the latter allows to extend the definition to singular cardinals.
From classical to generalized descriptive set theory:

**DST:**
- Cantor space $\omega^2 \sim \lambda$-Cantor space $\lambda^2$
- Baire space $\omega_\omega \sim \lambda$-Baire space $\text{cf}(\lambda)\lambda$
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V. Dimonte, L. Motto Ros and X. Shi, forthcoming paper on GDST on singular cardinals of countable cofinality.
Motivations and goals

**Aim:** study GDST on $\lambda$ singular of uncountable cofinality.

**What we want:**
A suitable class $\lambda$-DST of Polish-like spaces of weight $\lambda$ that:

1. includes $\lambda^2$ and $\text{cf}(\lambda)\lambda$.
2. can support most of DST tools and results.
3. for $\lambda = \omega$ gives exactly Polish spaces.
4. goes well with different definitions of $\lambda$-Polish for other known cases.

**Context:** $T_3$ (regular and Hausdorff) topological spaces, cardinals $\lambda$ satisfying $2^{<\lambda} = \lambda$. 
What is known: $\lambda$ singular

Why should we want to study these spaces for $\lambda$ singular?

Lambda singular recovers parts of classical DST that "fail" (or simply are much different/harder) in GDST on $\kappa$ regular.

$\lambda$ singular of countable cofinality: much can be recovered (PSP $\Sigma^1_1$, Silver Dichotomy, ...) (V. Dimonte, L. Motto Ros and X. Shi, forthcoming)

Definition

Let $\lambda$ be a (singular) cardinal of countable cofinality. A $\lambda$-Polish space is a completely metrizable space of weight $\lambda$.

Remark

The $\lambda$-Cantor and $\lambda$-Baire spaces are metrizable if and only if $\text{cf}(\lambda) = \omega$. 

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**Theorem**

Let $X$ be a second countable $(T_1, \text{regular})$ space. Then

- $X$ is metrizable.
- $X$ is Polish if and only if $X$ is strong Choquet.
What is known: \( \lambda \) regular

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The *strong Choquet game* on \( X \) is played in the following way:

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<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>I</td>
<td>( V_0, x_0 )</td>
<td>( V_1, x_1 )</td>
</tr>
<tr>
<td>II</td>
<td>( U_0 )</td>
<td>( U_1 )</td>
</tr>
</tbody>
</table>

- \( V_\alpha \) and \( U_\alpha \) are nonempty (if possible) open sets.
- \( V_\alpha \subseteq U_\beta \subseteq V_\gamma \) for every \( \gamma \leq \beta < \alpha < \omega \).
- \( x_\alpha \in V_\alpha \) and \( x_\alpha \in U_\alpha \) for every \( \alpha < \omega \).

The first player \( I \) wins if \( \bigcap_{\alpha < \omega} U_\alpha = \emptyset \), otherwise \( II \) wins.
What is known: $\lambda$ regular

**Theorem**

Let $X$ be a second countable ($T_1$, regular) space. Then

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The *strong $\delta$-Choquet game* on $X$ is played in the following way:

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<tr>
<th>I</th>
<th>$V_0, x_0$</th>
<th>$V_1, x_1$</th>
<th>...</th>
<th>$V_\gamma, x_\gamma$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$U_0$</td>
<td>$U_1$</td>
<td>...</td>
<td>$U_\gamma$</td>
<td>...</td>
</tr>
</tbody>
</table>

- $V_\alpha$ and $U_\alpha$ are nonempty (if possible) relatively open sets.
- $V_\alpha \subseteq U_\beta \subseteq V_\gamma$ for every $\gamma \leq \beta < \alpha < \delta$.
- $x_\alpha \in V_\alpha$ and $x_\alpha \in U_\alpha$ for every $\alpha < \delta$.

The first player I wins if $\bigcap_{\alpha<\delta} U_\alpha = \emptyset$, otherwise II wins.
What is known: $\lambda$ regular

Coskey and Schlicht, *Generalized choquet spaces*, 2016:
Let $\kappa$ be a regular cardinal. The class of strong $\kappa$-Choquet spaces has desirable properties for GDST.

Remark
Let $\lambda$ be a singular cardinal. There are strong $\lambda$-Choquet topological spaces of weight $\lambda$ with "patological" behaviour.

What goes wrong?
For $\lambda$ regular the spaces preserve some properties of metric spaces that are not preserved for $\lambda$ singular.
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### Restoring metrizability

<table>
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<th>Polish</th>
<th>$\lambda$-DST</th>
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<tr>
<td>Second countability</td>
<td>$\sim$  $\lambda$-DST weight $\lambda$</td>
</tr>
<tr>
<td>Completeness</td>
<td>$\sim$  strong $\text{cf}(\lambda)$-Choquet</td>
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<tr>
<td>Metrizability</td>
<td>$\sim$  ?</td>
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*Theorem (Nagata-Smirnov metrization theorem)*

Let $X$ be a topological space. Then $X$ is metrizable if and only if $X$ admits a $\sigma$-locally finite base.

*Definition*

Let $X$ be a topological space, and $A$ a family of subsets of $X$.

- We say $A$ is locally finite if every point $x \in X$ has a neighborhood $U$ intersecting finitely many pieces of $A$.
- We say $A$ is $\sigma$-locally finite if it has a cover $A = \bigcup_{i \in \omega} A_i$ of countable size such that each $A_i$ is locally finite.
Theorem (Nagata-Smirnov metrization theorem)

Let $X$ be a topological space. Then $X$ is metrizable if and only if $X$ admits a \( \sigma \)-locally finite base.

Definition

Let $X$ be a topological space, and $\mathcal{A}$ a family of subsets of $X$. We say $\mathcal{A}$ is locally finite if every point $x \in X$ has a neighborhood $U$ intersecting finitely many pieces of $\mathcal{A}$. We say $\mathcal{A}$ is \( \sigma \)-locally finite if it has a cover $\mathcal{A} = \bigcup_{i \in \omega} \mathcal{A}_i$ of countable size such that each $\mathcal{A}_i$ is locally finite.
Restoring metrizability

Polish

Second countability

Completeness

Metrizability

$\lambda$-DST

$\lambda$-DST weight $\lambda$

Strong $\text{cf}(\lambda)$-Choquet

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Let $X$ be a topological space. Then $X$ is metrizable if and only if $X$ admits a $\sigma$-locally finite base.

## Definition

Let $X$ be a topological space, and $\mathcal{A}$ a family of subsets of $X$. We say $\mathcal{A}$ is **locally $\gamma$-small** if every point $x \in X$ has a neighborhood $U$ intersecting $< \gamma$ many pieces of $\mathcal{A}$. We say $\mathcal{A}$ is **$\gamma$-Nagata-Smirnov** if it has a cover $\mathcal{A} = \bigcup_{i \in \gamma} A_i$ of size $\gamma$ such that each $A_i$ is locally $\gamma$-small.
Theorem (Nagata-Smirnov metrization theorem)

Let $X$ be a topological space. Then $X$ is metrizable if and only if $X$ admits a $\omega$-Nagata-Smirnov base.

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Definition
Let $\lambda$ be a cardinal. We call $\lambda$-DST a strong $\text{cf}(\lambda)$-Choquet topological space of weight $\lambda$ with a $\text{cf}(\lambda)$-Nagata-Smirnov base.
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Every base of size $\lambda$ is $\lambda$-Nagata-Smirnov: it can be covered by $\lambda$ many singletons.

Proposition

Let $\lambda$ be a cardinal.

- If $\lambda$ regular, $\lambda$-DST means strong $\lambda$-Choquet of weight $\lambda$. 
Definition

Let \( \lambda \) be a cardinal. We call \( \lambda \)-DST a strong cf(\( \lambda \))-Choquet topological space of weight \( \lambda \) with a cf(\( \lambda \))-Nagata-Smirnov base.

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- If \( \lambda = \omega \), \( \lambda \)-DST means Polish.
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- If $\lambda$ regular, $\lambda$-DST means strong $\lambda$-Choquet of weight $\lambda$.
- If $\lambda = \omega$, $\lambda$-DST means Polish.
- If $\lambda$ uncountable of countable cofinality, $\lambda$-DST means completely metrizable of weight $\lambda$. (proof to be checked)
Examples and non-examples

Examples of $\lambda$-DST spaces:

1. The $\lambda$-Cantor and $\lambda$-Baire spaces.
2. Completely metrizable spaces of weight $\lambda$.
3. For every tree $T$ of density $\lambda$ and uniform height, $[T]$ with the bounded topology is $\lambda$-DST.
4. If $X$ is $\lambda$-DST, then $\mathcal{K}(X)$ with the Vietoris topology is $\lambda$-DST.
5. Disjoint unions of $\lambda$-many $\lambda$-DST spaces are $\lambda$-DST.
6. Products of $\text{cf}(\lambda)$-many $\lambda_i$-DST spaces are $\sup(\lambda_i)$-DST.
7. Open subspaces of a $\lambda$-DST are $\lambda$-DST.

Non-examples:

1. Products of $> \text{cf}(\lambda)$ many non-trivial spaces are never $\lambda$-DST.
2. If $\text{cf}(\lambda) > \omega$, there is a closed subspace of $\lambda^2$ which is not $\lambda$-DST.
3. If $\text{cf}(\lambda) > \omega$, there is a $\lambda$-DST space whose perfect part is not $\lambda$-DST.
Some results

How much can we restore of classical descriptive set theory?
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**Theorem ([2, Theorem 7.9])**

Let $X$ be a Polish space. There is a continuous surjective function $f : \omega \omega \to X$ and a closed $C \subseteq \omega \omega$ such that $f \upharpoonright C$ is bijective.
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**Theorem (A., Motto Ros)**

Let $X$ be a $\lambda$-DST space. There is a continuous surjective function $f : \text{cf}(\lambda) \lambda \to X$ and a closed $C \subseteq \text{cf}(\lambda) \lambda$ such that $f \upharpoonright C$ is bijective.
We can get more:

**Theorem (A., Motto Ros)**

Suppose $X$ is a $\text{cf}(\lambda)$-additive $\lambda$-DST space and $\text{cf}(\lambda) > \omega$. Then $X$ is homeomorphic to a (super)closed subspace of $\text{cf}(\lambda) \lambda$.

(needs some cardinal assumption if $\lambda$-singular)

**Recall**: $X$ is $\gamma$ additive if the intersection of $< \gamma$ open sets is open.

**Recall**: $C$ superclosed if $C = [T]$ for $T$ homogeneous in height.

**Recall**: A tree $T$ is homogeneous in height if every branch has same height.
Theorem ([2, Theorem 6.2])

Let \( X \) be a prefect Polish space. There is an embedding of \( \omega^2 \) into \( X \).
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Let $X$ be a prefect Polish space. There is an embedding of $\omega_2$ into $X$.

Coskey, Schlicht [1]: let $X$ be $\lambda$-perfect, strong $\lambda$-Choquet for $\lambda$ regular. There is a continuous injective function from $\lambda^2$ into $X$.

V. Dimonte, L. Motto Ros, X. Shi: let $X$ be $\lambda$-perfect $\lambda$-Polish space. There is an embedding of $\lambda^2$ into $X$ with closed image.

**Definition:** $X$ $\lambda$-perfect if no intersection of $< \text{cf}(\lambda)$ opens has size $< \lambda$. 
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Theorem (A., Motto Ros)

Let $X$ be a $\lambda$-perfect $\lambda$-DST space. There is a continuous injective function from $^{\lambda}2$ into $X$ with $\lambda$-Borel inverse.

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If there exists $A \subseteq \text{cf}(\lambda) \lambda$ without the Perfect Set Property, then there exists a $\lambda$-DST subset $B \subseteq \text{cf}(\lambda) \lambda$ without the PSP.

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Super $\lambda$-Choquet game:

- Same game as before, but players can play only big open sets (of size $\lambda$).

Super $\lambda$-DST:

- Super $\lambda$-Choquet game instead of strong $\lambda$-Choquet.

**Theorem (A., Motto Ros)**

Let $X$ be a $\lambda$-DST space. Then the perfect kernel of $X$ is $\lambda$-DST if and only if $X$ is super $\lambda$-DST.

**Corollary**

Let $X$ be super $\lambda$-DST. Then either $X$ divides $\lambda$ or there is a continuous injective function from $2^{\lambda}$ into $X$. 

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$\lambda$-DST spaces 

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**Corollary**

*Let $X$ be super $\lambda$-DST. Then $|X| \leq \lambda$ or there is a continuous injective function from $\lambda^2$ into $X$.*
S. Coskey and P. Schlicht.
Generalized choquet spaces.

A. Kechris.
*Classical descriptive set theory*, volume 156.