A category-theoretic framework for Fraïssé theory

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Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories. We say that an $\mathcal{L}$-object $U$ is

- **universal or cofinal** in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object $X$ there is an $\mathcal{L}$-map $X \to U$,

- **homogeneous** in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-maps $f, g : X \to U$ from a $\mathcal{K}$-object there is an $\mathcal{L}$-automorphism $h : U \to U$ such that $f = h \circ g$,

- **injective** (or that it has the extension property) in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map $f : X \to U$ from a $\mathcal{K}$-object and every $\mathcal{K}$-map $g : X \to Y$ there is an $\mathcal{L}$-map $h : Y \to U$ such that $f = h \circ g$.

Observation

A universal homogeneous object is injective, but a universal injective object may not be homogeneous.
Applications – Classical Fraïssé theory

- The ambient category consists of all structures and all embeddings of a fixed first-order language.
- $\mathcal{K}$ is a full subcategory of some finitely generated structures.
- $\mathcal{L}$ is the full subcategory of all unions of increasing chains of $\mathcal{K}$-objects.

<table>
<thead>
<tr>
<th>$\mathcal{K}$</th>
<th>$\mathcal{L}$</th>
<th>universal homogeneous object in $\langle \mathcal{K}, \mathcal{L} \rangle$</th>
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<tr>
<td>finite linear orders</td>
<td>countable linear orders</td>
<td>the rationals</td>
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<td>finite graphs</td>
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<td>finite groups</td>
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<td>finite rational metric spaces</td>
<td>countable rational metric spaces</td>
<td>rational Urysohn space</td>
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Applications – Projective Fraïssé theory [Irwin–Solecki, 2006]

- A *topological structure* is a first-order structure endowed with a compact Hausdorff zero-dimensional topology such that the functions are continuous and the relations are closed.

- A *quotient map* of topological structures is a continuous surjective homomorphism such that every satisfied relation in the codomain has a witness in the domain.

- The ambient category is the opposite category to the category consisting of all topological structures and all quotient maps of a fixed first-order language.

- $\mathcal{K}$ is a subcategory whose objects are some finite structures.

- $\mathcal{L}$ is the category of limits of sequences in $\mathcal{K}$ (the sequences are inverse sequences of quotient maps).
Applications – Projective Fraïssé theory [Irwin–Solecki, 2006]

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<td>Menger curve pre-space</td>
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Introduction

How to obtain a universal homogeneous object?

1. Start with a sufficiently nice category $\mathcal{K}$, so it is possible to build a Fraïssé sequence.

2. Observe that it is possible to interpret the Fraïssé sequence as a universal homogeneous object in the category of sequences.

3. If $\mathcal{K}$ is nicely placed in a larger category, then we may move from sequences to their limits – the limit of the Fraïssé sequence is a universal and homogeneous object.
1. Fraïssé sequences

Theorem

A category $\mathcal{K} \neq \emptyset$ has a Fraïssé sequence if and only if the following conditions hold:

1. $\mathcal{K}$ has a countable dominating subcategory,
2. $\mathcal{K}$ is directed,
3. $\mathcal{K}$ has the amalgamation property.

Definition

We will call such category a *Fraïssé category*. 
1. Fraïssé sequences

Definition

Let $\mathcal{K}$ be a category.

- $\mathcal{K}$ is countable if there are only countably many $\mathcal{K}$-maps.

- $\mathcal{K}$ is directed if for every two $\mathcal{K}$-objects $X, Y$ there are $\mathcal{K}$-maps $f : X \rightarrow W$, $g : Y \rightarrow W$ to a common codomain.

- $\mathcal{K}$ has the amalgamation property (AP) if for every $\mathcal{K}$-maps $f : Z \rightarrow X$, $g : Z \rightarrow Y$ from a common domain there are $\mathcal{K}$-maps $f' : X \rightarrow W$, $g' : Y \rightarrow W$ to a common codomain such that $f' \circ f = g' \circ g$. 
1. Fraïssé sequences

By a *sequence* in $\mathcal{K}$ we mean a direct sequence $\langle X_*, f_* \rangle$, i.e.
- $X_* = \langle X_n \rangle_{n \in \omega}$ is a sequence of $\mathcal{K}$-objects,
- $f_* = \langle f_n : X_n \to X_{n+1} \rangle_{n \in \omega}$ is a sequence of $\mathcal{K}$-maps.

The sequence may have a (co)limit $\langle X_\infty, f_*^\infty \rangle$, where
- $X_\infty$ is the limit object, and
- $f_*^\infty = \langle f_n^\infty : X_n \to X_\infty \rangle_{n \in \omega}$ is the limit cone.

$$
X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots X_n \xrightarrow{f_n} X_{n+1} \to \cdots X_\infty
$$
Definition

Let $S$ be a subcategory of $\mathcal{K}$ or a sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$.

- $S$ is **cofinal** in $\mathcal{K}$ if for every $\mathcal{K}$-object $X$ there is a $\mathcal{K}$-map $f : X \to Y$ to an $S$-object.

- $S$ is **absorbing** in $\mathcal{K}$ if for every $\mathcal{K}$-map $f$ from an $S$-object there is a $\mathcal{K}$-map $f'$ such that $f' \circ f$ is an $S$-map.
  In the sequence case, $\text{dom}(f) = X_n$ for a fixed $n$ and $f' \circ f$ has to be $f^m_n$ for some $m \geq n$.

- $S$ is **injective** in $\mathcal{K}$ if for every $\mathcal{K}$-maps $f, g$ from a common domain and with $\text{cod}(f)$ being an $S$-object there exist an $S$-map $f'$ and a $\mathcal{K}$-map $g'$ such that $f' \circ f = g' \circ g$.
  In the sequence case, $\text{cod}(f) = X_n$ for a fixed $n$ and $f'$ has to be $f^m_n$ for some $m \geq n$.

- $S$ is **dominating** in $\mathcal{K}$ if it is cofinal and absorbing in $\mathcal{K}$.

- $S$ is **Fraïssé** in $\mathcal{K}$ if it is cofinal and injective in $\mathcal{K}$. 
1. Fraïssé sequences

**Figure:** Implications between the properties of $S$ in $\mathcal{K}$. 
1. Fraïssé sequences

**Theorem**

A category $\mathcal{K} \neq \emptyset$ has a Fraïssé sequence if and only if the following conditions hold:

1. $\mathcal{K}$ has a countable dominating subcategory,
2. $\mathcal{K}$ is directed,
3. $\mathcal{K}$ has the amalgamation property.

**Definition**

We will call such category a *Fraïssé category*. 
2. Categories of sequences

Theorem

Let $\mathcal{K}$ be a category and let $\langle X_*, f_* \rangle$ be a sequence in $\mathcal{K}$. The following conditions are equivalent.

1. $\langle X_*, f_* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. $\langle X_*, f_* \rangle$ is a universal and injective object in $\langle \mathcal{K}, \sigma_0 \mathcal{K} \rangle$.
3. $\langle X_*, f_* \rangle$ is a universal and homogeneous object in $\langle \mathcal{K}, \sigma_0 \mathcal{K} \rangle$.

Moreover, a sequence satisfying the conditions is unique up to isomorphism in $\sigma_0 \mathcal{K}$, and it is universal in $\langle \sigma_0 \mathcal{K}, \sigma_0 \mathcal{K} \rangle$. 
Definition

- A **transformation** \( \langle F_*, \varphi \rangle : \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle \) between sequences in \( \mathcal{K} \) is a pair \( \langle F_*, \varphi \rangle \) such that
  - \( \varphi : \omega \to \omega \) is an increasing cofinal map, and
  - \( F_* = \langle F_n : X_n \to Y_{\varphi(n)} \rangle_{n \in \omega} \) is a sequence of \( \mathcal{K} \)-maps such that
    \[
    g_{\varphi(n)}^m \circ F_n = F_m \circ f_n^m \quad \text{for every } n \leq m \in \omega,
    \]
i.e. it is a natural transformation from \( \langle X_*, f_* \rangle \) to \( \langle Y_*, g_* \rangle \circ \varphi \).

- \( \text{Seq} (\mathcal{K}) \) denotes the category of all sequences in \( \mathcal{K} \) and all transformations between them.

- Two transformations \( \langle F_*, \varphi \rangle, \langle G_*, \psi \rangle : \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle \) are **equivalent** if for every \( n \in \omega \) we have \( F_n \approx g_* \circ G_n \), i.e. there is \( m \geq \varphi(n), \psi(n) \) such that
  \[
  g_{\varphi(n)}^m \circ F_n = g_{\psi(n)}^m \circ G_n.
  \]
We write \( \langle F_*, \varphi \rangle \approx \langle G_*, \psi \rangle \).

- The relation \( \approx \) is a congruence on the category \( \text{Seq} (\mathcal{K}) \).
  \( \sigma_0 \mathcal{K} \) denotes the quotient category \( \text{Seq} (\mathcal{K}) / \approx \).
2. Categories of sequences

Let $J: \mathcal{K} \to \text{Seq}(\mathcal{K})$ be the functor that assigns to every $\mathcal{K}$-object $X$ the constant sequence $\langle \langle X \rangle_{n \in \omega}, \langle \text{id}_X \rangle_{n \in \omega} \rangle$, and to every $\mathcal{K}$-map $f: X \to Y$ the constant transformation $\langle \langle f \rangle_{n \in \omega}, \text{id}_\omega \rangle$.

- A $\sigma_0\mathcal{K}$-map $\langle F_*, \varphi \rangle: \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle$ is determined by $\mathcal{K}$-maps $F_{n_k}: X_{n_k} \to Y_{\varphi(n_k)}$ such that $F_{n_k} \approx g_* \circ F_{n_{k+1}} \circ f_{n_k}^{n_{k+1}}$ for an increasing sequence $\langle n_k \rangle_{k \in \omega}$.

- A $\sigma_0\mathcal{K}$-map $J(X) \to \langle Y_*, g_* \rangle$ is determined by a $\mathcal{K}$-map $f: X \to Y_n$ for some $n$.

- A $\sigma_0\mathcal{K}$-map $J(X) \to J(Y)$ is determined by a unique $\mathcal{K}$-map $f: X \to Y$, so $J: \mathcal{K} \to \sigma_0\mathcal{K}$ is a full embedding, and we may identify $\mathcal{K}$ with the full subcategory of $\sigma_0\mathcal{K}$ consisting of constant sequences.

- For every sequence $\mathcal{X}$ in $\sigma_0\mathcal{K}$, the diagonal sequence in $\mathcal{K}$ is the limit of $\mathcal{X}$ in $\sigma_0\mathcal{K}$. In particular, every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$ is its own limit in $\sigma_0\mathcal{K}$. So we have constructed $\sigma_0\mathcal{K}$ essentially by adding formal limits of sequences in $\mathcal{K}$. 

2. Categories of sequences

Proposition (back and forth)

Let \( \langle X_*, f_* \rangle \) and \( \langle Y_*, g_* \rangle \) be sequences in \( \mathcal{K} \).

1. If the sequences are absorbing, then every \( \mathcal{K} \)-map \( F_{n_0} : X_{n_0} \to Y_{m_0} \) can be extended to a \( \sigma_0 \mathcal{K} \)-isomorphism \( F_* : \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle \).

2. If the sequences are injective, then for every \( \mathcal{K} \)-maps \( F : Z \to X_n \) and \( G : Z \to Y_m \) there is a \( \sigma_0 \mathcal{K} \)-isomorphism \( H_* : \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle \) such that \( G \approx_{g_*} H_n \circ F \).

Corollary

Fraïssé sequences are unique up to \( \sigma_0 \mathcal{K} \)-isomorphism.

Corollary

An injective sequence in \( \mathcal{K} \) is a homogeneous object in \( \langle \mathcal{K}, \sigma_0 \mathcal{K} \rangle \).
2. Categories of sequences

Theorem

Let $\mathcal{K}$ be a category and let $\langle X_*, f_* \rangle$ be a sequence in $\mathcal{K}$. The following conditions are equivalent.

1. $\langle X_*, f_* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. $\langle X_*, f_* \rangle$ is a universal and injective object in $\langle \mathcal{K}, \sigma_0 \mathcal{K} \rangle$.
3. $\langle X_*, f_* \rangle$ is a universal and homogeneous object in $\langle \mathcal{K}, \sigma_0 \mathcal{K} \rangle$.

Moreover, a sequence satisfying the conditions is unique up to isomorphism in $\sigma_0 \mathcal{K}$, and it is universal in $\langle \sigma_0 \mathcal{K}, \sigma_0 \mathcal{K} \rangle$. 
3. Nice extensions

\[ \mathcal{K} \text{ is often a subcategory of a larger category } \mathcal{L} \text{ such that sequences in } \mathcal{K} \text{ have limits in } \mathcal{L}. \text{ In that case, we want to move from sequences to their limits and consider the corresponding category } \sigma \mathcal{K} \subseteq \mathcal{L}. \]

**Theorem**

Let \( \mathcal{K} \) be a *nicely placed* subcategory of \( \mathcal{L} \). For every sequence \( \langle \mathcal{X}_*, f_* \rangle \) in \( \mathcal{K} \) the following conditions are equivalent.

1. \( \langle \mathcal{X}_*, f_* \rangle \) is a Fraïssé sequence in \( \mathcal{K} \).
2. \( \mathcal{X}_\infty \) is a universal and injective object in \( \langle \mathcal{K}, \sigma \mathcal{K} \rangle \).
3. \( \mathcal{X}_\infty \) is a universal and homogeneous object in \( \langle \mathcal{K}, \sigma \mathcal{K} \rangle \).

Moreover, such \( \sigma \mathcal{K} \)-object \( \mathcal{X}_\infty \) is unique up to isomorphism, and it is universal in \( \langle \sigma \mathcal{K}, \sigma \mathcal{K} \rangle \).
Let $\mathcal{K} \subseteq \mathcal{L}$ be categories such that sequences in $\mathcal{K}$ have limits in $\mathcal{L}$.

- For every Seq($\mathcal{K}$)-map $\langle F_*, \varphi \rangle: \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle$ and every choice of limit cones $\langle X_\infty, f_\infty \rangle, \langle Y_\infty, g_\infty \rangle$ there is a unique $\mathcal{L}$-map $F_\infty: X_\infty \to Y_\infty$ such that $g_{\varphi(n)} \circ F_n = F_\infty \circ f_{\infty}$ for every $n \in \omega$ – we shall call it the limit of the transformation.

- This assignment defines a limit functor $L: \text{Seq}(\mathcal{K}) \to \mathcal{L}$. The functor factorizes through $\approx$, and hence also $L: \sigma_0 \mathcal{K} \to \mathcal{L}$.

- By $\sigma \mathcal{K}$ we denote the subcategory of $\mathcal{L}$ generated by limits of all transformations of sequences in $\mathcal{K}$ for all choices of their limit cones.
3. Nice extensions

Let $\mathcal{K} \subseteq \mathcal{L}$ and let $L : \sigma_0 \mathcal{K} \to \sigma \mathcal{K}$ be a limit functor. Let us consider the following conditions.

(L1) For every $\mathcal{K}$-maps $f : X \to Y_n$, $f' : X \to Y_{n'}$ from a $\mathcal{K}$-object $X$ to a sequence $\langle Y_*, g_* \rangle$ in $\mathcal{K}$ such that $g_n^\infty \circ f = g_{n'}^\infty \circ f'$ there exists $m \geq n, n'$ such that $g_m^n \circ f = g_{n'}^m \circ f'$.

(L2) For every sequence $\langle Y_*, g_* \rangle$ in $\mathcal{K}$ and every $\sigma \mathcal{K}$-map $f : X \to Y_\infty$ from a $\mathcal{K}$-object there exists a $\mathcal{K}$-map $f' : X \to Y_n$ such that $g_n^\infty \circ f' = f$.

**Proposition**

1. (L1) $\iff$ $L$ is “faithful from small” $\iff$ $L$ is faithful.

2. (L2) $\iff$ $L$ is “full from small” $\iff$ $L$ is full $\iff$ (L1) & (L2).
3. Nice extensions

**Definition**

\( \mathcal{K} \) is *nicely placed* in \( \mathcal{L} \) if \( \mathcal{K} \subseteq \mathcal{L} \), every sequence in \( \mathcal{K} \) has a limit in \( \mathcal{L} \), and \( \langle \mathcal{K}, \mathcal{L} \rangle \) satisfies (L1) and (L2).

**Observation**

1. If \( \mathcal{K} \) is nicely placed in \( \mathcal{L} \), then any limit functor \( L: \sigma_0 \mathcal{K} \to \sigma \mathcal{K} \) is an equivalence of categories.
2. (L1) holds if \( \sigma \mathcal{K} \) consists of monomorphisms.
3. (L2) holds if and only if there is a \( \sigma_0 \mathcal{K} \)-isomorphism \( F_*: \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle \) with \( F_\infty = \text{id} \) whenever \( X_\infty = Y_\infty \).
4. In the classical model-theoretical setting and in the projective Fraïssé theory, the conditions (L1) and (L2) are satisfied.
3. Nice extensions

Let us recall the main result of this section.

**Theorem**

Let $\mathcal{K}$ be a *nicely placed* subcategory of $\mathcal{L}$. For every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$ the following conditions are equivalent.

1. $\langle X_*, f_* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. $X_\infty$ is a universal and injective object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$.
3. $X_\infty$ is a universal and homogeneous object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$.

Moreover, such $\sigma\mathcal{K}$-object $X_\infty$ is unique up to isomorphism, and it is universal in $\langle \sigma\mathcal{K}, \sigma\mathcal{K} \rangle$.

**Remark**

If $\sigma\mathcal{K}$ is a full subcategory of $\mathcal{L}$, then a universal homogeneous object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ is also universal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$, but universal homogeneous objects in $\langle \mathcal{K}, \mathcal{L} \rangle$ are not unique in general.
A summarizing definition

We say that an \( \mathcal{L} \)-object \( X \) is a *Fraïssé limit* of \( \mathcal{K} \) in \( \mathcal{L} \), and we write \( X = \text{Flim}_\mathcal{L}(\mathcal{K}) \), if \( \mathcal{K} \) is nicely placed in \( \mathcal{L} \) and \( X \) satisfies the following equivalent conditions:

- \( X \) is universal and homogeneous in \( \langle \mathcal{K}, \mathcal{L} \rangle \);
- \( X \) is universal and injective in \( \langle \mathcal{K}, \mathcal{L} \rangle \);
- \( X \) is a limit in \( \mathcal{L} \) of a Fraïssé sequence in \( \mathcal{K} \).

Necessarily, \( \mathcal{K} \) is a Fraïssé category.
The presented framework can be extended in at least three orthogonal ways:

1. beyond the countable case – when uncountable sequences or directed diagrams are considered,
2. by weakening the amalgamation property – which is closely connected with the abstract Banach–Mazur game,
3. beyond the discrete case – when the strict commutativity of diagrams is replaced by $\varepsilon$-commutativity with better and better $\varepsilon$ (in the metric-enriched setting).