A category-theoretic framework for Fraïssé theory

Adam Bartoš
drekin@gmail.com

Institute of Mathematics, University of Wrocław

Winter School in Abstract Analysis
Section Set Theory & Topology
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Introduction

Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories. We say that an $\mathcal{L}$-object U is

- *universal* or *cofinal* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object X there is an $\mathcal{L}$-map $X \to U$,
- *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-maps $f, g \colon X \to U$ from a $\mathcal{K}$-object there is an $\mathcal{L}$-automorphism $h \colon U \to U$ such that $f = h \circ g$,
- *injective* (or that it has the *extension property*) in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map $f \colon X \to U$ from a $\mathcal{K}$-object and every $\mathcal{K}$-map $g \colon X \to Y$ there is an $\mathcal{L}$-map $h \colon Y \to U$ such that $f = h \circ g$.

Observation

A universal homogeneous object is injective, but a universal injective object may not be homogeneous.

Introduction

Applications – Classical Fraïssé theory

- The ambient category consists of all structures and all embeddings of a fixed first-order language.
- $\mathcal{K}$ is a full subcategory of some finitely generated structures.
- $\mathcal{L}$ is the full subcategory of all unions of increasing chains of $\mathcal{K}$-objects.

$\mathcal{K}$	$\mathcal{L}$	universal homogeneous object in $\langle \mathcal{K}, \mathcal{L} \rangle$
finite linear orders	countable linear orders	the rationals
finite graphs	countable graphs	Rado graph
finite groups	locally finite countable groups	Hall's universal group
finite rational metric spaces	countable rational metric spaces	rational Urysohn space

Introduction

Applications – Projective Fraïssé theory [Irwin–Solecki, 2006]

- A *topological structure* is a first-order structure endowed with a compact Hausdorff zero-dimensional topology such that the functions are continuous and the relations are closed.
- A *quotient map* of topological structures is a continuous surjective homomorphism such that every satisfied relation in the codomain has a witness in the domain.
- The ambient category is the opposite category to the category consisting of all topological structures and all quotient maps of a fixed first-order language.
- $\mathcal{K}$ is a subcategory whose objects are some finite structures.
- $\mathcal{L}$ is the category of limits of sequences in $\mathcal{K}$ (the sequences are inverse sequences of quotient maps).

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Applications – Projective Fraïssé theory [Irwin–Solecki, 2006]

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$\mathcal{K}$	universal homogeneous object in $\langle \mathcal{K}, \mathcal{L} \rangle$
finite connected linear graphs and all quotients	pseudo-arc pre-space [Irwin–Solecki, 2006]
finite connected graphs and connected quotients	Menger curve pre-space [Panagiotopoulos–Solecki, 2019]

Introduction

How to obtain a universal homogeneous object?

1. Start with a sufficiently nice category $\mathcal{K}$, so it is possible to build a *Fraïssé sequence*.
2. Observe that it is possible to interpret the Fraïssé sequence as a universal homogeneous object in the *category of sequences*.
3. If $\mathcal{K}$ is *nicely placed* in a larger category, then we may move from sequences to their limits – the limit of the Fraïssé sequence is a universal and homogeneous object.

1. Fraïssé sequences

Theorem

A category $\mathcal{K} \neq \emptyset$ has a Fraïssé sequence if and only if the following conditions hold:

1. $\mathcal{K}$ has a countable dominating subcategory,
2. $\mathcal{K}$ is directed,
3. $\mathcal{K}$ has the amalgamation property.

Definition

We will call such category a *Fraïssé category*.

1. Fraïssé sequences

Definition

Let $\mathcal{K}$ be a category.

- $\mathcal{K}$ is *countable* if there are only countably many $\mathcal{K}$-maps.
- $\mathcal{K}$ is *directed* if for every two $\mathcal{K}$-objects X, Y there are $\mathcal{K}$-maps $f\colon X \to W$, $g\colon Y \to W$ to a common codomain.
- $\mathcal{K}$ has the *amalgamation property (AP)* if for every $\mathcal{K}$-maps $f\colon Z \to X$, $g\colon Z \to Y$ from a common domain there are $\mathcal{K}$-maps $f'\colon X \to W$, $g'\colon Y \to W$ to a common codomain such that $f' \circ f = g' \circ g$.

By a *sequence* in $\mathcal{K}$ we mean a direct sequence $\langle X_*, f_* \rangle$, i.e.

- $X_* = \langle X_n \rangle_{n\in\omega}$ is a sequence of $\mathcal{K}$-objects,
- $f_* = \langle f_n \colon X_n \to X_{n+1} \rangle_{n\in\omega}$ is a sequence of $\mathcal{K}$-maps.

The sequence may have a (co)limit $\langle X_\infty, f_*^\infty \rangle$, where

- X_∞ is the limit object, and
- $f_*^\infty = \langle f_n^\infty \colon X_n \to X_\infty \rangle_{n\in\omega}$ is the limit cone.

$$X_0 \xrightarrow{f_0} X_1 \underbrace{\xrightarrow{f_1} X_2 \xrightarrow{f_2}}_{f_1^3} X_3 \xrightarrow{f_3} \cdots X_n \underbrace{\xrightarrow{f_n} X_{n+1} \to \cdots}_{f_n^\infty} X_\infty$$

1. Fraïssé sequences

Definition

Let $\mathcal{S}$ be a subcategory of $\mathcal{K}$ or a sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$.

- $\mathcal{S}$ is *cofinal* in $\mathcal{K}$ if for every $\mathcal{K}$-object X there is a $\mathcal{K}$-map $f \colon X \to Y$ to an $\mathcal{S}$-object.
- $\mathcal{S}$ is *absorbing* in $\mathcal{K}$ if for every $\mathcal{K}$-map f from an $\mathcal{S}$-object there is a $\mathcal{K}$-map f' such that $f' \circ f$ is an $\mathcal{S}$-map.
  In the sequence case, $\mathrm{dom}(f) = X_n$ for a fixed n and $f' \circ f$ has to be f_n^m for some $m \geq n$.
- $\mathcal{S}$ is *injective* in $\mathcal{K}$ if for every $\mathcal{K}$-maps f, g from a common domain and with $\mathrm{cod}(f)$ being an $\mathcal{S}$-object there exist an $\mathcal{S}$-map f' and a $\mathcal{K}$-map g' such that $f' \circ f = g' \circ g$.
  In the sequence case, $\mathrm{cod}(f) = X_n$ for a fixed n and f' has to be f_n^m for some $m \geq n$.
- $\mathcal{S}$ is *dominating* in $\mathcal{K}$ if it is cofinal and absorbing in $\mathcal{K}$.
- $\mathcal{S}$ is *Fraïssé* in $\mathcal{K}$ if it is cofinal and injective in $\mathcal{K}$.

1. Fraïssé sequences

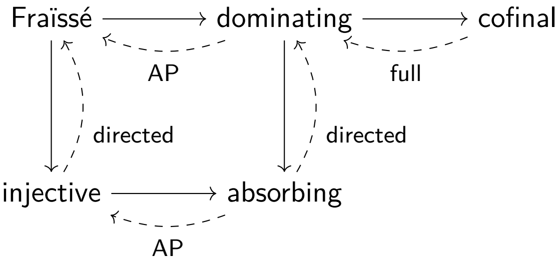

Figure: Implications between the properties of $\mathcal{S}$ in $\mathcal{K}$.

Theorem

A category $\mathcal{K} \neq \emptyset$ has a Fraïssé sequence if and only if the following conditions hold:

1. $\mathcal{K}$ has a countable dominating subcategory,
2. $\mathcal{K}$ is directed,
3. $\mathcal{K}$ has the amalgamation property.

Definition

We will call such category a *Fraïssé category*.

Theorem

Let $\mathcal{K}$ be a category and let $\langle X_*, f_* \rangle$ be a sequence in $\mathcal{K}$.
The following conditions are equivalent.

1. $\langle X_*, f_* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. $\langle X_*, f_* \rangle$ is a universal and injective object in $\langle \mathcal{K}, \sigma_0\mathcal{K} \rangle$.
3. $\langle X_*, f_* \rangle$ is a universal and homogeneous object in $\langle \mathcal{K}, \sigma_0\mathcal{K} \rangle$.

Moreover, a sequence satisfying the conditions is unique up to isomorphism in $\sigma_0\mathcal{K}$, and it is universal in $\langle \sigma_0\mathcal{K}, \sigma_0\mathcal{K} \rangle$.

2. Categories of sequences

Definition

- A *transformation* $\langle F_*, \varphi \rangle \colon \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle$ between sequences in $\mathcal{K}$ is a pair $\langle F_*, \varphi \rangle$ such that
  - $\varphi \colon \omega \to \omega$ is an increasing cofinal map, and
  - $F_* = \langle F_n \colon X_n \to Y_{\varphi(n)} \rangle_{n \in \omega}$ is a sequence of $\mathcal{K}$-maps such that $g_{\varphi(n)}^{\varphi(m)} \circ F_n = F_m \circ f_n^m$ for every $n \leq m \in \omega$,

  i.e. it is a natural transformation from $\langle X_*, f_* \rangle$ to $\langle Y_*, g_* \rangle \circ \varphi$.
- $\mathrm{Seq}(\mathcal{K})$ denotes the category of all sequences in $\mathcal{K}$ and all transformations between them.
- Two transformations $\langle F_*, \varphi \rangle, \langle G_*, \psi \rangle \colon \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle$ are *equivalent* if for every $n \in \omega$ we have $F_n \approx_{g*} G_n$,
  i.e. there is $m \geq \varphi(n), \psi(n)$ such that $g_{\varphi(n)}^m \circ F_n = g_{\psi(n)}^m \circ G_n$.
  We write $\langle F_*, \varphi \rangle \approx \langle G_*, \psi \rangle$.
- The relation $\approx$ is a congruence on the category $\mathrm{Seq}(\mathcal{K})$.
  $\sigma_0\mathcal{K}$ denotes the quotient category $\mathrm{Seq}(\mathcal{K})/\approx$.

2. Categories of sequences

Let $J\colon \mathcal{K} \to \mathrm{Seq}(\mathcal{K})$ be the functor that assigns to every $\mathcal{K}$-object X the constant sequence $\langle\langle X\rangle_{n\in\omega}, \langle \mathrm{id}_X\rangle_{n\in\omega}\rangle$, and to every $\mathcal{K}$-map $f\colon X \to Y$ the constant transformation $\langle\langle f\rangle_{n\in\omega}, \mathrm{id}_\omega\rangle$.

- A $\sigma_0\mathcal{K}$-map $\langle F_*, \varphi\rangle\colon \langle X_*, f_*\rangle \to \langle Y_*, g_*\rangle$ is determined by $\mathcal{K}$-maps $F_{n_k}\colon X_{n_k} \to Y_{\varphi(n_k)}$ such that $F_{n_k} \approx_{g_*} F_{n_{k+1}} \circ f_{n_k}^{n_{k+1}}$ for an increasing sequence $\langle n_k\rangle_{k\in\omega}$.
- A $\sigma_0\mathcal{K}$-map $J(X) \to \langle Y_*, g_*\rangle$ is determined by a $\mathcal{K}$-map $f\colon X \to Y_n$ for some n.
- A $\sigma_0\mathcal{K}$-map $J(X) \to J(Y)$ is determined by a unique $\mathcal{K}$-map $f\colon X \to Y$, so $J\colon \mathcal{K} \to \sigma_0\mathcal{K}$ is a full embedding, and we may identify $\mathcal{K}$ with the full subcategory of $\sigma_0\mathcal{K}$ consisting of constant sequences.
- For every sequence $\mathcal{X}$ in $\sigma_0\mathcal{K}$, the diagonal sequence in $\mathcal{K}$ is the limit of $\mathcal{X}$ in $\sigma_0\mathcal{K}$. In particular, every sequence $\langle X_*, f_*\rangle$ in $\mathcal{K}$ is its own limit in $\sigma_0\mathcal{K}$. So we have constructed $\sigma_0\mathcal{K}$ essentially by adding formal limits of sequences in $\mathcal{K}$.

2. Categories of sequences

Proposition (back and forth)

Let $\langle X_*, f_* \rangle$ and $\langle Y_*, g_* \rangle$ be sequences in $\mathcal{K}$.

1. If the sequences are absorbing, then every $\mathcal{K}$-map $F_{n_0}\colon X_{n_0} \to Y_{m_0}$ can be extended to a $\sigma_0\mathcal{K}$-isomorphism $F_*\colon \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle$.
2. If the sequences are injective, then for every $\mathcal{K}$-maps $F\colon Z \to X_n$ and $G\colon Z \to Y_m$ there is a $\sigma_0\mathcal{K}$-isomorphism $H_*\colon \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle$ such that $G \approx_{g_*} H_n \circ F$.

Corollary

Fraïssé sequences are unique up to $\sigma_0\mathcal{K}$-isomorphism.

Corollary

An injective sequence in $\mathcal{K}$ is a homogeneous object in $\langle \mathcal{K}, \sigma_0\mathcal{K} \rangle$.

2. Categories of sequences

Theorem

Let $\mathcal{K}$ be a category and let $\langle X_*, f_* \rangle$ be a sequence in $\mathcal{K}$.
The following conditions are equivalent.

1. $\langle X_*, f_* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. $\langle X_*, f_* \rangle$ is a universal and injective object in $\langle \mathcal{K}, \sigma_0 \mathcal{K} \rangle$.
3. $\langle X_*, f_* \rangle$ is a universal and homogeneous object in $\langle \mathcal{K}, \sigma_0 \mathcal{K} \rangle$.

Moreover, a sequence satisfying the conditions is unique up to isomorphism in $\sigma_0 \mathcal{K}$, and it is universal in $\langle \sigma_0 \mathcal{K}, \sigma_0 \mathcal{K} \rangle$.

3. Nice extensions

$\mathcal{K}$ is often a subcategory of a larger category $\mathcal{L}$ such that sequences in $\mathcal{K}$ have limits in $\mathcal{L}$. In that case, we want to move from sequences to their limits and consider the corresponding category $\sigma\mathcal{K} \subseteq \mathcal{L}$.

Theorem

Let $\mathcal{K}$ be a *nicely placed* subcategory of $\mathcal{L}$. For every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$ the following conditions are equivalent.

1. $\langle X_*, f_* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. X_∞ is a universal and injective object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$.
3. X_∞ is a universal and homogeneous object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$.

Moreover, such $\sigma\mathcal{K}$-object X_∞ is unique up to isomorphism, and it is universal in $\langle \sigma\mathcal{K}, \sigma\mathcal{K} \rangle$.

3. Nice extensions

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories such that sequences in $\mathcal{K}$ have limits in $\mathcal{L}$.

- For every Seq($\mathcal{K}$)-map $\langle F_*, \varphi\rangle \colon \langle X_*, f_*\rangle \to \langle Y_*, g_*\rangle$ and every choice of limit cones $\langle X_\infty, f_*^\infty\rangle$, $\langle Y_\infty, g_*^\infty\rangle$ there is a unique $\mathcal{L}$-map $F_\infty \colon X_\infty \to Y_\infty$ such that $g_{\varphi(n)}^\infty \circ F_n = F_\infty \circ f_n^\infty$ for every $n \in \omega$ – we shall call it the *limit of the transformation*.
- This assignment defines a *limit functor* $L\colon$ Seq($\mathcal{K}$) $\to \mathcal{L}$. The functor factorizes through $\approx$, and hence also $L\colon \sigma_0\mathcal{K} \to \mathcal{L}$.
- By $\sigma\mathcal{K}$ we denote the subcategory of $\mathcal{L}$ generated by limits of all transformations of sequences in $\mathcal{K}$ for all choices of their limit cones.

3. Nice extensions

Let $\mathcal{K} \subseteq \mathcal{L}$ and let $L\colon \sigma_0\mathcal{K} \to \sigma\mathcal{K}$ be a limit functor. Let us consider the following conditions.

(L1) For every $\mathcal{K}$-maps $f\colon X \to Y_n$, $f'\colon X \to Y_{n'}$ from a $\mathcal{K}$-object X to a sequence $\langle Y_*, g_* \rangle$ in $\mathcal{K}$ such that $g_n^\infty \circ f = g_{n'}^\infty \circ f'$ there exists $m \geq n, n'$ such that $g_n^m \circ f = g_{n'}^m \circ f'$.

(L2) For every sequence $\langle Y_*, g_* \rangle$ in $\mathcal{K}$ and every $\sigma\mathcal{K}$-map $f\colon X \to Y_\infty$ from a $\mathcal{K}$-object there exists a $\mathcal{K}$-map $f'\colon X \to Y_n$ such that $g_n^\infty \circ f' = f$.

Proposition

1. (L1) $\iff$ L is "faithful from small" $\iff$ L is faithful.
2. (L2) $\iff$ L is "full from small" $\Longleftarrow$ L is full $\Longleftarrow$ (L1) & (L2).

3. Nice extensions

Definition

$\mathcal{K}$ is *nicely placed* in $\mathcal{L}$ if $\mathcal{K} \subseteq \mathcal{L}$, every sequence in $\mathcal{K}$ has a limit in $\mathcal{L}$, and $\langle \mathcal{K}, \mathcal{L} \rangle$ satisfies (L1) and (L2).

Observation

1. If $\mathcal{K}$ is nicely placed in $\mathcal{L}$, then any limit functor $L \colon \sigma_0\mathcal{K} \to \sigma\mathcal{K}$ is an equivalence of categories.
2. (L1) holds if $\sigma\mathcal{K}$ consists of monomorphisms.
3. (L2) holds if and only if there is a $\sigma_0\mathcal{K}$-isomorphism $F_* \colon \langle X_*, f_* \rangle \to \langle Y_*, g_* \rangle$ with $F_\infty = \mathrm{id}$ whenever $X_\infty = Y_\infty$.
4. In the classical model-theoretical setting and in the projective Fraïssé theory, the conditions (L1) and (L2) are satisfied.

3. Nice extensions

Let us recall the main result of this section.

Theorem

Let $\mathcal{K}$ be a *nicely placed* subcategory of $\mathcal{L}$. For every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$ the following conditions are equivalent.

1. $\langle X_*, f_* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. X_∞ is a universal and injective object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$.
3. X_∞ is a universal and homogeneous object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$.

Moreover, such $\sigma\mathcal{K}$-object X_∞ is unique up to isomorphism, and it is universal in $\langle \sigma\mathcal{K}, \sigma\mathcal{K} \rangle$.

Remark

If $\sigma\mathcal{K}$ is a full subcategory of $\mathcal{L}$, then a universal homogeneous object in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ is also universal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$, but universal homogeneous objects in $\langle \mathcal{K}, \mathcal{L} \rangle$ are not unique in general.

A summarizing definition

We say that an $\mathcal{L}$-object X is a *Fraïssé limit* of $\mathcal{K}$ in $\mathcal{L}$, and we write $X = \mathrm{Flim}_{\mathcal{L}}(\mathcal{K})$, if $\mathcal{K}$ is nicely placed in $\mathcal{L}$ and X satisfies the following equivalent conditions:

- X is universal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$;
- X is universal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$;
- X is a limit in $\mathcal{L}$ of a Fraïssé sequence in $\mathcal{K}$.

Necessarily, $\mathcal{K}$ is a Fraïssé category.

Generalizations

The presented framework can be extended in at least three orthogonal ways:

1. beyond the countable case – when uncountable sequences or directed diagrams are considered,
2. by weakening the amalgamation property – which is closely connected with the abstract Banach–Mazur game,
3. beyond the discrete case – when the strict commutativity of diagrams is replaced by ε-commutativity with better and better ε (in the metric-enriched setting).