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Reaping and Rosenthal Families

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Rosenthal's lemma

Rosenthal's lemma

Let $\mathcal{A}$ be a Boolean algebra, $\{\mu_k : \mathcal{A} \to \mathbb{R}_+ \cup \{0\}\}_{k\in\omega}$ be a uniformly bounded sequence of finitely additive measures on $\mathcal{A}$ and let $(A_n)_{n\in\omega}$ be pairwise disjoint elements of $\mathcal{A}$ and $\varepsilon > 0$. Then there is a $A \in \mathcal{R} = [\omega]^\omega$ such that for every $k \in A$ we have

$$\sum_{n\in A\setminus\{k\}} \mu_k(A_n) \le \varepsilon.$$

Introduction

A set $D \subseteq [\omega]^\omega$ is *dense* if for all $A \in [\omega]^\omega$ there is $B \in D$ such that $B \subseteq A$. Given a non-empty collection $\mathcal{D}$ of dense sets, a set $G \subseteq [\omega]^\omega$ is $\mathcal{D}$-*generic* if $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

Definition

Given a non-empty collection $\mathcal{D}$ of dense sets

$$\mathfrak{gen}(\mathcal{D}) = \min\{|G| : G \text{ is } \mathcal{D}-\textit{generic}\}$$

More definitions

A matrix $M = (m_{i,j})_{i,j\in\omega}$ is a *Rosenthal matrix* if

- $m_{i,j} \geq 0$,
- $m_{i,i} = 0$,
- $||M|| < \infty$,

where $||M|| = \sup\{\sum_{j\in\omega} m_{i,j} : i \in \omega\}$.

If $A \subseteq \omega$, then $M \upharpoonright A = (m_{i,j})_{i,j\in A}$.

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If $A \subseteq \omega$, then $M \restriction A = (m_{i,j})_{i,j\in A}$.

Rosenthal lemma

For every Rosenthal matrix M and for every $\varepsilon > 0$ the set

$$D_{M,\varepsilon} = \{A \in [\omega]^\omega : ||M \restriction A|| \leq \varepsilon\}$$

is dense.

Rosenthal families

Let

$$\mathcal{R} = \{D_{M,\varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix }\}$$

Definition [D. Sobota]

A Rosenthal family is a generic for $\mathcal{R}$, $\mathfrak{ros} = \mathfrak{gen}(\mathcal{R})$.

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Questions

- Selective Ultrafilters are Rosenthal Families. Is it true that all ultrafilters are Rosenthal families?
- What is the value of $\mathfrak{ros}$?

Reaping families

A family $R \subseteq [\omega]^\omega$ is *reaping* if for any partition of $\omega = A \cup B$, there is a $C \in R$ such that either $C \subseteq A$ or $C \subseteq B$

$$\mathfrak{r} = \min\{|R| : R \text{ is a reaping family }\}.$$

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A non-empty family $R \subseteq [\omega]^\omega$ is *hereditaly reaping* if for every $A \in R$, the family $\{B \in R : B \subseteq A\}$ is reaping.

$$\mathfrak{r} = \min\{|R| : R \text{ is a hereditarily reaping family }\}.$$

Paving theorem

Theorem (Bourgain)

If M is a Rosenthal matrix and $\varepsilon > 0$, then there is a partition of $\omega = A_0 \cup \ldots \cup A_k$ into finitely many pieces such that for every $i \in 0, \ldots, i$, $||M \upharpoonright A_i|| \leq \varepsilon$.

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If M is a Rosenthal matrix and $\varepsilon > 0$, then there is a partition of $\omega = A_0 \cup \ldots \cup A_k$ into finitely many pieces such that for every $i \in 0, \ldots, i$, $||M \restriction A_i|| \leq \varepsilon$.

Given any partition into finitely many pieces, any hereditary reaping family will have to pick a piece, so

Corollary

Ultrafilter $\Rightarrow$ hereditarily reaping $\Rightarrow$ Rosenthal family.

$$\mathfrak{ros} \leq \mathfrak{r}$$

Nowhere reaping

A family $R \subseteq [\omega]^\omega$ is *nowhere reaping* if for every $A \in [\omega]^\omega$, the family $\{A \cap B : B \in R \text{ and } A \cap B \text{ is infinite}\}$ is not reaping.

Families with less than $\mathfrak{r}$ elements are nowhere reaping.

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Theorem

Rosenthal families are somewhere reaping (not nowhere reaping).

$$\mathfrak{r} = \mathfrak{ros}$$

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Questions

- Are all Rosenthal families reaping families?
- Are all Rosenthal filters ultrafilters?

Rosenthal matrices as functions

If M is a Rosenthal matrix, then

$$\begin{array}{rccc} F_M\colon & c_0 & \to & \ell_\infty \\ & \overline{x} & & M\overline{x} \end{array}$$

is a continuous linear function such that $||F_M||_\infty = ||M||$

And basically, every bounded linear function from c_0 to ℓ_∞ looks like this.

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$$D_{T,\varepsilon} = \{A \in [\omega]^\omega : ||P_A \cdot T \cdot P_A||_\infty \leq \varepsilon \cdot ||T||_\infty\}$$

$$\mathcal{R}(X, Y) = \{D_{T,\varepsilon} : \varepsilon > 0, T \in \mathcal{B}(X, Y)\}$$

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Theorem

Rosenthal families are exactly the generic families for $\mathcal{R}(c_0, \ell_\infty)$.

Rosenthal families and Banach spaces

Recall

$$\mathcal{R} = \{D_{M,\varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix}\}$$

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Consider

$$\mathcal{R}_1 = \{D_{M,\varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix}$$
$$\text{whose columns converge to } 0\}$$

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$$\text{whose columns converge to } 0\}$$

Theorem

$\mathcal{R}_1$ generic families are exactly the generic families for $\mathcal{R}(c_0, c_0)$ and $\mathfrak{gen}(\mathcal{R}_1) = \min\{\mathfrak{r}, \mathfrak{d}\} = \mathfrak{ros}(c_0)$.

Free sets

One important class of Rosenthal matrices M are the ones of only 1s and 0s, which can be coded by functions f_M from $\omega \to \omega$ without fixed points. In this case, a set A has the property $||M \restriction A|| < \frac{1}{2}$ if and only if $f_M(A) \cap A = \emptyset$.

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$$\mathfrak{ros} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ s.t. } \forall f : \omega \to \omega \text{ w.o. fixed points}$$

$$\exists A \in \mathcal{A} \text{ s.t. } f(A) \cap A = \emptyset\}$$

$$\mathfrak{ros}(c_0) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ s.t. } \forall f : \omega \to \omega \text{ finite to one w.o. fixed points}$$

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Question

What is the corresponding cardinal invariant for the injective case?

Cardinal Invariants in specific Banach spaces

Theorem

Rosenthal families are exactly generic families for $\mathcal{R}(\ell_1, \ell_1)$. As a consequence $\mathfrak{ros}(\ell_1) = \mathfrak{r}$.

Question

What can be said about the rest of the ℓ_p?

Theorem

Rosenthal families are exactly generic families for $\mathcal{R}(\ell_1, \ell_1)$. As a consequence $\mathfrak{ros}(\ell_1) = \mathfrak{r}$.

Question

What can be said about the rest of the ℓ_p?

Theorem (Kadison-Singer problem)[Marcus, Spielman, Srivastava]

For every $\varepsilon > 0$ and every $T \in \mathcal{B}(\ell_2)$ there is a finite partition $\omega = A_0, \ldots, A_n$ such that for every $i \in 0, \ldots, n$, $||P_A T P_A||_\infty \leq \varepsilon \cdot ||T||_\infty$. Therefore $\mathfrak{ros}(\ell_2) \leq \mathfrak{r}$.

Such theorem is impossible for $p = \infty$.

Thank you for your attention!

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References

[1] Piotr Koszmider and Arturo Martínez-Celis. Rosenthal families, pavings and generic cardinal invariants. *arXiv/1911.01336*, 2019.