Reaping and Rosenthal Families

Arturo Martínez-Celis
Joint work with Piotr Koszmider[1]
January 2020.
Hejnice, Czech Republic.

Winter School in Abstract Analysis 2020.
Rosenthal’s lemma

Let $\mathcal{A}$ be a Boolean algebra, $\{\mu_k : \mathcal{A} \to \mathbb{R}_+ \cup \{0\}\}_{k \in \omega}$ be a uniformly bounded sequence of finitely additive measures on $\mathcal{A}$ and let $(A_n)_{n \in \omega}$ be pairwise disjoint elements of $\mathcal{A}$ and $\varepsilon > 0$. Then there is a $A \in \mathcal{R} = [\omega]^{\omega}$ such that for every $k \in A$ we have

$$\sum_{n \in A \setminus \{k\}} \mu_k(A_n) \leq \varepsilon.$$
A set $D \subseteq [\omega]^{\omega}$ is dense if for all $A \in [\omega]^{\omega}$ there is $B \in D$ such that $B \subseteq A$. Given a non-empty collection $\mathcal{D}$ of dense sets, a set $G \subseteq [\omega]^{\omega}$ is $\mathcal{D}$-generic if $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

**Definition**

Given a non-empty collection $\mathcal{D}$ of dense sets

$$\text{gen}(\mathcal{D}) = \min\{|G| : G \text{ is } \mathcal{D} \text{-generic}\}$$
A matrix $M = (m_{i,j})_{i,j \in \omega}$ is a Rosenthal matrix if

- $m_{i,j} \geq 0$,
- $m_{i,i} = 0$,
- $||M|| < \infty$,

where $||M|| = \sup\{\sum_{j \in \omega} m_{i,j} : i \in \omega\}$.

If $A \subseteq \omega$, then $M \upharpoonright A = (m_{i,j})_{i,j \in A}$.

Rosenthal lemma

For every Rosenthal matrix $M$ and for every $\varepsilon > 0$ the set $D_M;\varepsilon = \{A \subseteq \omega : jjM^\upharpoonright A jj < \varepsilon\}$ is dense.
More definitions

A matrix $M = (m_{i,j})_{i,j \in \omega}$ is a Rosenthal matrix if

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where $\|M\| = \sup \{ \sum_{j \in \omega} m_{i,j} : i \in \omega \}$.

If $A \subseteq \omega$, then $M \upharpoonright A = (m_{i,j})_{i,j \in A}$.

**Rosenthal lemma**

For every Rosenthal matrix $M$ and for every $\varepsilon > 0$ the set

$$D_{M,\varepsilon} = \{ A \in [\omega]^\omega : \|M \upharpoonright A\| \leq \varepsilon \}$$

is dense.
Let

\[ \mathcal{R} = \{ D_{M,\varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix} \} \]

Definition [D. Sobota]
A Rosenthal family is a generic for \( \mathcal{R} \), \( \text{ros} = \text{gen}(\mathcal{R}) \).
Rosenthal families

Let

\[ \mathcal{R} = \{ D_{M, \varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix} \} \]

**Definition [D. Sobota]**

A Rosenthal family is a generic for \( \mathcal{R} \), \( \text{ros} = \text{gen}(\mathcal{R}) \).

**Questions**

- Selective Ultrafilters are Rosenthal Families. Is it true that all ultrafilters are Rosenthal families?
- What is the value of \( \text{ros} \)?
A family $R \subseteq [\omega]^{\omega}$ is *reaping* if for any partition of $\omega = A \cup B$, there is a $C \in R$ such that either $C \subseteq A$ or $C \subseteq B$.

$$r = \min\{|R| : R \text{ is a reaping family}\}.$$
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A non-empty family $R \subseteq [\omega]^\omega$ is hereditarily reaping if for every $A \in R$, the family $\{B \in R : B \subseteq A\}$ is reaping.

$$r = \min\{|R| : R \text{ is a hereditarily reaping family }\}.$$
### Theorem (Bourgain)

If $M$ is a Rosenthal matrix and $\varepsilon > 0$, then there is a partition of $\omega = A_0 \cup \ldots \cup A_k$ into finitely many pieces such that for every $i \in 0, \ldots, i$, $\|M \restriction A_i\| \leq \varepsilon$. 

**Corollary**

Ultrafilter (hereditarily reaping) Rosenthal family.
Theorem (Bourgain)

If $M$ is a Rosenthal matrix and $\varepsilon > 0$, then there is a partition of $\omega = A_0 \cup \ldots \cup A_k$ into finitely many pieces such that for every $i \in 0, \ldots, k$, $\|M \upharpoonright A_i\| \leq \varepsilon$.

Given any partition into finitely many pieces, any hereditary reaping family will have to pick a piece, so

Corollary

Ultrafilter $\Rightarrow$ hereditarily reaping $\Rightarrow$ Rosenthal family.

$\text{ros} \leq \mathfrak{r}$
Nowhere reaping

A family $R \subseteq [\omega]^\omega$ is nowhere reaping if for every $A \in [\omega]^\omega$, the family \(\{A \cap B : B \in R \text{ and } A \cap B \text{ is infinite}\}\) is not reaping.

Families with less than \(\aleph_1\) elements are nowhere reaping.
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Families with less than $r$ elements are nowhere reaping.

**Theorem**

Rosenthal families are somewhere reaping (not nowhere reaping).

$$r = \text{ros}$$
Nowhere reaping

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Families with less than \( r \) elements are nowhere reaping.

**Theorem**

Rosenthal families are somewhere reaping (not nowhere reaping).

\[ r = \text{ros} \]

**Questions**

- Are all Rosenthal families reaping families?
- Are all Rosenthal filters ultrafilters?
If $M$ is a Rosenthal matrix, then

$$F_M: \quad c_0 \rightarrow \ell_\infty$$

$$M \bar{x}$$

is a continuous linear function such that $\|F_M\|_\infty = \|M\|$

And basically, every bounded linear function from $c_0$ to $\ell_\infty$ looks like this.
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$$D_{T, \varepsilon} = \{ A \in [\omega]^\omega : ||P_A \cdot T \cdot P_A||_\infty \leq \varepsilon \cdot ||T||_\infty \}$$

$$\mathcal{R}(X, Y) = \{ D_{T, \varepsilon} : \varepsilon > 0, T \in \mathcal{B}(X, Y) \}$$
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**Theorem**

Rosenthal families are exactly the generic families for $\mathcal{R}(c_0, \ell_\infty)$. 
Recall

\[ \mathcal{R} = \{ D_{M,\varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix} \} \]

Consider

\[ \mathcal{R}_1 = \{ D_{M,\varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix whose columns converge to 0} \} \]
Recall

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Consider

\[ \mathcal{R}_1 = \{ D_{M,\varepsilon} : \varepsilon > 0, M \text{ is a Rosenthal matrix} \text{ whose columns converge to 0} \} \]

**Theorem**

\( \mathcal{R}_1 \) generic families are exactly the generic families for \( \mathcal{R}(c_0, c_0) \) and \( \text{gen}(\mathcal{R}_1) = \min\{r, d\} = \text{ros}(c_0) \).
Free sets

One important class of Rosenthal matrices $M$ are the ones of only 1s and 0s, which can be coded by functions $f_M$ from $\omega \to \omega$ without fixed points. In this case, a set $A$ has the property $||M \upharpoonright A|| < \frac{1}{2}$ if and only if $f_M(A) \cap A = \emptyset$. 

Question

What is the corresponding cardinal invariant for the injective case?
Free sets

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$$\text{ros} = \min\{|A| : A \subseteq [\omega]^\omega \text{ s.t. } \forall f : \omega \to \omega \text{ w.o. fixed points}$$

$$\exists A \in \mathcal{A} \text{ s.t. } f(A) \cap A = \emptyset \}$$

$$\text{ros}(c_0) = \min\{|A| : A \subseteq [\omega]^\omega \text{ s.t. } \forall f : \omega \to \omega \text{ finite to one w.o. fixed points}$$

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Theorem
Rosenthal families are exactly generic families for $R(\ell_1, \ell_1)$. As a consequence $\text{ros}(\ell_1) = r$.

Question
What can be said about the rest of the $\ell_p$?

Theorem (Kadison-Singer problem)[Marcus, Spielman, Srivastava]
For every $\varepsilon > 0$ and every $T \in B(\ell_2)$ there is a finite partition $\omega = A_0, \ldots, A_n$ such that for every $i \in 0, \ldots, n$, $\|P_A TP_A\|_\infty \leq \varepsilon \cdot \|T\|_\infty$. Therefore $\text{ros}(\ell_2) \leq r$.

Such theorem is impossible for $p = \infty$. 
Thank you for your attention!

arodriguez@impan.pl