Infinite partitions on free products of two Boolean algebras

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Definition (Free product)

If A and B are two boolean algebras, their free product, $A \oplus B$ is an algebra C such that $A, B \leq C$,

$$C = \langle A \cup B \rangle := \left\{ \sum_{i<n} a_i \cdot b_i \mid n < \omega, a_i \in A, b_i \in B \right\}$$

and for all $a \in A \setminus \{0\}$ and all $b \in B \setminus \{0\}$, $a \cdot b \neq 0$.

Topologically if $A \cong clop\,(X)$ and $B \cong clop\,(Y)$, then $A \oplus B \cong clop\,(X \times Y)$.

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Partitions

Let A be an infinite boolean algebra.

Definition

If $X \subseteq A \setminus \{0\}$ such that $a \cdot b = 0$ for all $a, b \in X$, and such that for all $c \in A \setminus \{0\}$ there exists $a \in X$ such that $a \cdot c \neq 0$, it will be said that X is a *partition* of A.

Definition (Partition number)

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Centered families

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$X \subseteq A$ is said to be a *centered family* if for all $F \in [X]^{<\omega} \setminus \{\emptyset\}$, $0 \neq \prod F$.

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If $p \in A \setminus \{0\}$ is such that $p \leq x$ for all $x \in X$, a centered family, it will be said that p is a *pseudointersection* of X.

Definition (Pseudointersection number)

$\mathfrak{p}(A) :=$
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Observation

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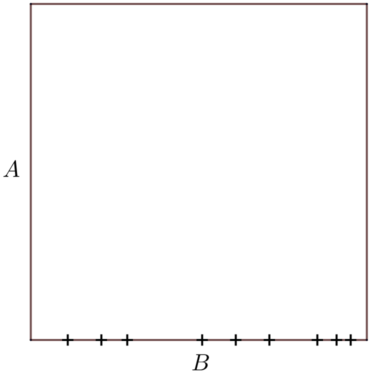
Theorem

Let A and B be two infinite boolean algebras. Then $\mathfrak{a}(A \oplus B) \leq min\{\mathfrak{a}(A), \mathfrak{a}(B)\}$



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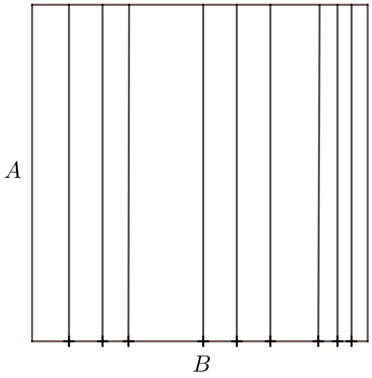

The natural question to ask

Given A and B infinite boolean algebras
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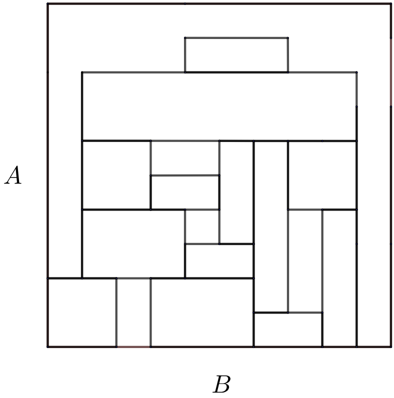

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$$\mathfrak{a}\left(A \oplus B\right) \geq min\left\{min\left\{\mathfrak{a}\left(A\right), \mathfrak{a}\left(B\right)\right\}, max\left\{\mathfrak{p}\left(A\right), \mathfrak{p}\left(B\right)\right\}\right\}$$

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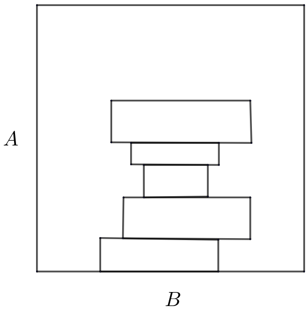
Let κ be a cardinal less than $\mathfrak{p}(B)$ and $\mathfrak{a}(A)$ and take $\{a_\alpha \cdot b_\alpha \mid \alpha < \kappa\}$, a disjoint family on the free product. Case 1: $\forall E \in [\kappa]^{\geq\omega}\, \exists F \in [E]^{<\omega}\, \forall \alpha \in E\ b_\alpha \leq \sum_{\beta \in F} b_\beta$. As an easy consequence, there exists an infinite (maximal) centered family of b_α.



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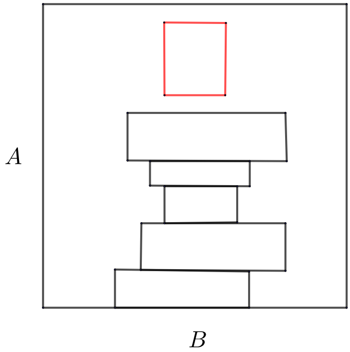
We take b, a pseudointersection of these b_α's and a an element disjoint to these a_α's and we are done.



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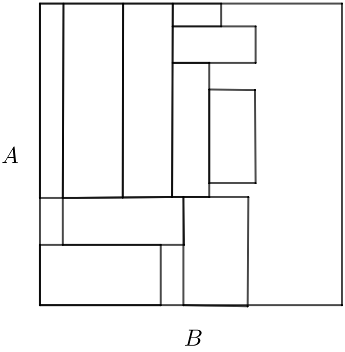
Case 2: $\exists E \in [\kappa]^{\geq\omega} \forall F \in [E]^{<\omega} \exists \alpha \in E\ b_\alpha \nleq \sum_{\beta \in F} b_\beta$. Let it be maximal.



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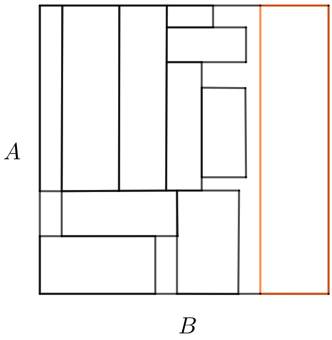
There is $b \in B$ which is not covered by b_α with $\alpha \in E$.



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If $A = P(\omega)/Fin = B$, previous theorem says that $\mathfrak{p} \leq \mathfrak{a}(P(\omega)/Fin \oplus P(\omega)/Fin) \leq \mathfrak{a}$

Theorem

$min\{\mathfrak{a}, \mathfrak{s}\} \leq \mathfrak{a}(P(\omega)/Fin \oplus P(\omega)/Fin)$

This theorem also holds for any pair of homogeneous boolean algebras.

Question

Is it possible that $\mathfrak{s} = \mathfrak{a}(P(\omega)/Fin \oplus P(\omega)/Fin) < \mathfrak{a}$? (Hechler?)

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