Fréchet-like properties and ad families

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UNAM-UMSNH

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Fréchet, $\alpha_i$ and strong Fréchet properties

A point $x \in X$ is a **Fréchet point** if whenever $x \in \overline{A}$ there is a sequence $\{x_n : n \in \omega\} \subseteq A$ such that $x_n \rightarrow x$.

**Definition (Arhangel’skii, 79)**

A point $x \in X$ is an $\alpha_i$-point ($i = 1, 2, 3, 4$) if given a family $\{S_n : n \in \omega\}$ of sequences converging to $x$, there is a sequence $S \rightarrow x$ (we identify a convergent sequence with its range) such that:

1. $(\alpha_1)$ $S \setminus S_n$ is finite for all $n \in \omega$,
2. $(\alpha_2)$ $S \cap S_n \neq \emptyset$ for all $n \in \omega$,
3. $(\alpha_3)$ $|S \cap S_n| = \omega$ for infinitely many $n \in \omega$,
4. $(\alpha_4)$ $S \cap S_n \neq \emptyset$ for infinitely many $n \in \omega$.

Then a space $X$ is Fréchet (resp. $\alpha_i$) if every point $x \in X$ is Fréchet (resp. $\alpha_i$).
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A space $X$ is **absolutely Fréchet** if in some Hausdorff compactification $bX$ of $X$, every point $x \in X$ is a Fréchet point.

Given a family $\mathcal{A} \subseteq \mathcal{P}(X)$ we will say that $x \in \overline{\mathcal{A}}$ ($x$ clusters at $\mathcal{A}$) if $x \in \overline{A}$ for every $A \in \mathcal{A}$. A filter base $\mathcal{G}$ **converges** to a point $x \in X$ if for every neighborhood $V$ of $x$, there is a $G \in \mathcal{G}$ such that $G \subseteq V$. We then write $\mathcal{G} \rightarrow x$.

**Definition (E. Michael, 72)**

$X$ is **bisequential** at $x \in X$ if for every ultrafilter $\mathcal{U}$ in $X$ such that $x \in \overline{\mathcal{U}}$ there is a sequence $\mathcal{G} = \{G_n : n \in \omega\} \subseteq \mathcal{U}$ such that $\mathcal{G} \rightarrow x$. A space $X$ is bisequential if it is bisequential at every point.
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Relationship between strong Fréchet and $\alpha_1$ properties

First countable

$\alpha_1$

$\alpha_2$

bisequential

$\alpha_3$

absolutely Fréchet

$\alpha_4$

Fréchet
Impact of these properties in the product of Fréchet spaces

- The product of a countably compact space with an $\alpha_3$-FU space is Fréchet.
- $X \times [0, 1]$ is Fréchet iff $X$ is $\alpha_4$-FU.
- If $X$ is absolutely Fréchet and $Y$ is first countable then $X \times Y$ is absolutely Fréchet.
- The product of an absolutely Fréchet and a bisequential space is absolutely Fréchet.
Some results

- (Arhangel’skii, 79) There is a Fréchet space which is not $\alpha_4$.
- (Arhangel’skii, 79) There is a non-bisequential space $X$ such that it is $\alpha_4$-FU.
- (Simon, 80) There is an $\alpha_4$ space which is not $\alpha_3$.
- (Nyikos, 89) There is an $\alpha_3$ space which is not $\alpha_2$.
- (Nyikos, 90’s) There is an $\alpha_2$ space that is not first countable.
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A family \( \mathcal{A} \subseteq [\omega]^\omega \) is an almost disjoint (ad) family if \( |A \cap B| < \omega \) for every \( A, B \in \mathcal{A} \). \( \mathcal{A} \) is maximal almost disjoint (mad) if it is ad and maximal with respect to this property.

Given an ad family \( \mathcal{A} \), the ad space generated by \( \mathcal{A} \) is the subspace \( \omega \cup \{\infty\} \) of the one-point compactification of \( \Psi(\mathcal{A}) \).

We will say that an ad family \( \mathcal{A} \) satisfies a topological property \( P \) if its ad space does.

**Definition**

An ad family \( \mathcal{A} \) is hereditarily \( \alpha_3 \) if \( B \) is \( \alpha_3 \) for every \( B \subseteq \mathcal{A} \).

**Question (Gruenhage, 06)**

For an ad family \( \mathcal{A} \) is it equivalent being \( \alpha_3 \)-FU (hereditarily \( \alpha_3 \)-FU) with its bisequentiality?
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- (Folklore) There are bisequential ad families (actually, every $\mathbb{R}$-embedable is bisequential).
- (Nyikos, 09) Under $b = c$, there is a Fréchet ad family which is not $\alpha_3$. The example consists of graph of functions on $\omega \times \omega$, so...

Question (Nyikos, 09)
Is there a non-bisequential ad family consisting of functions such that it is $\alpha_3$-FU?

Theorem (C.-Hrušák)
non($\mathcal{M}$) = $c$. There exists an $\alpha_3$-FU (even $\alpha_2$-FU) ad family (consisting of functions in $\omega \times \omega$) which is not hereditarily $\alpha_3$. 
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\( b = c \). There exists an hereditarily \( \alpha_3 \)-FU almost disjoint family (consisting of partial functions) which is not bisequential.

Question

Can the above family consist of total functions?

Since \( b \leq \text{non}(\mathcal{M}) \), it follows that under \( b = c \) the tree concepts are different.
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Other constructions

**Theorem (C.-Hrušák)**

\((s \leq b)\) There is an \(\alpha_3\)-FU not hereditarily \(\alpha_3\) ad family.

**Corollary**

\((c \leq \aleph_2)\) There is an \(\alpha_3\)-FU non-bisequential ad family.

**Theorem (C.-Hrušák)**

\(\diamondsuit (b) \Rightarrow \) There is an \(\alpha_2\)-FU not her. \(\alpha_3\) ad family of size \(\omega_1\).

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- (79) Is there an absolutely Fréchet space which is not bisequential?
- (79) Is there a (countable) $\alpha_1$-FU space which is not bisequential?

A consistent example for the second question was given by Malyhin under the assumption $2^{\aleph_0} < 2^{\aleph_1}$.

**Theorem (C.-Hrušák)**

CH. There is a countable $\alpha_1$ and absolutely Fréchet space which is not bisequential.

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There is (in ZFC) an absolutely Fréchet space which is not bisequential.
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Some questions (if time allows)

- Is there (in ZFC) an $\alpha_3$-FU non-bisequential ad family?

Definition

Let $\mathcal{A}$ be an ad family:

- $\mathcal{A}$ is completely separable if for every $X \in \mathcal{I}(\mathcal{A})^+$ there exists $A \in \mathcal{A}$ such that $A \subseteq X$.

- $\mathcal{A}$ is almost completely separable if for every $X \subseteq \omega$ such that $X \cap A$ is infinite for infinitely many $A \in \mathcal{A}$, there exists $B \in \mathcal{A}$ such that $B \subseteq X$.

- $\mathcal{A}$ is weakly tight if for every family $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, there is $A \in \mathcal{A}$ such that $A \cap X_n$ is infinite for infinitely many $n \in \omega$.

- Is there (in ZFC) an almost weakly tight ad family?
- Does it follow from the existence of the above family, the existence of an $\alpha_3$-FU non-bisequential ad family?
Some questions (if time allows)

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Thank you for your attention!