The Uniform Subsets of the Eucliedean Plane

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Definition

Let $X$ be a subset of $\mathbb{R}^2$ and $\vec{e}$ is an arbitrary vector in $\mathbb{R}^2$, $X$ called an Uniform subset of $\mathbb{R}^2$ in direction $\vec{e}$ if for each $p'$ parallel to $\vec{e}$, we have

$$\text{card}(p' \cap X) \leq 1$$

Many years ago Luzin posed a problem, in particular Luzin asked whether there exists a function

$$\phi : \mathbb{R} \to \mathbb{R}$$

such that the whole plane $\mathbb{R}^2$ can be covered by countable many isometric copies of the graph of $\phi$. 
Partially, Sierpinski has answered to the Luzini Problem and under the Continuum Hypothesis has proved next theorem.

**Sierpinski’s Theorem.**

Assuming Continuum Hypothesis in $\mathbb{R}^2$ there exists two subsets $A$ and $B$, such that

1. The set $A$ is uniform with respect to the axis $\mathbb{R} \times 0$;
2. The set $B$ is uniform with respect to the axis $0 \times \mathbb{R}$;
3. There exists a countable family $\{h_n : n > \omega\}$ of translations of $\mathbb{R}^2$, for which we have

$$\bigcup \{h_n(A \cup B) : n < \omega\} = \mathbb{R}^2$$
Solution of Luzini Problem without CH

Theorem of Davies

Let \((\vec{e}_i)_{i \in \omega}\) be an injective countable family of vectors in \(\mathbb{R}^2\). Then there exists a family \(\{X_i : i \in \omega\}\) of subsets of \(\mathbb{R}^2\) such that

1. \(\bigcup\{X_i : k \in \omega\} = \mathbb{R}^2\);
2. for each \(i \in \omega\) the set \(X_i\) is uniform in direction \(\vec{e}_i\).
Finite set in direction vector $\vec{e}$

**Definition**

Let $\vec{e}$ be an arbitrary nonzero vector in $\mathbb{R}^2$. A set $B \subset \mathbb{R}^2$ is finite in direction $\vec{e}$ if

$$\text{card}(l \cap B) < \omega$$

for any straight line $l \subset \mathbb{R}^2$ parallel to $\vec{e}$.

**Theorem**

Let $Z \subset \mathbb{R}^2$ be a finite set in direction $\vec{e}$, where $\vec{e}$ is an arbitrary vector in the plane, then there exists an uniform set $X \subset \mathbb{R}^2$ in the same $\vec{e}$ direction, such that $Z$ is a countable many $\Pi_2$-configuration of $X$. 

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Let $E$ be a set and let $M$ be a class of measures on $E$ (in general, we do not assume that measures belonging to $M$ are defined on the one and same $\sigma$-algebra of subset of $E$).

**Definition**

- We say that a set $X \subset E$ is absolutely measurable with respect to $M$ if $X$ is measurable with respect to all measures from $M$.
- We say that a set $Y \subset E$ is relatively measurable with respect to $M$ if there exists at least one measure $\mu$ from $M$ such that $Y$ is $\mu$-measurable.
- We say that a set $Z \subset E$ is absolutely nonmeasurable with respect to $M$ if there exists no measure from $M$ such that $Z$ is measurable with respect to all measures from $M$. 
Measurability of the Uniform subset

Let $\Pi_2$ denote the group of all translations of the plane $\mathbb{R}^2$ and let $\lambda_2$ stand for the ordinary two-dimensional Lebesgue measure on $\mathbb{R}^2$.

**Theorem**

There exists a $\Pi_2$-invariant extension $\mu$ of the Lebesgue measure $\lambda_2$, such that all uniform sets in direction $Oy$-axis are measurable with respect $\mu$.

**Corollary.** The uniform set in any direction in $\mathbb{R}^2$ is absolutely measurable with respect to the class of all nonzero $\sigma$-finite $\Pi_2$-invariant measures.

**Theorem**

Under CH, there exist a set $A$ uniform in direction of $Oy$-axis and a set $B$ uniform in direction of $Ox$-axis, such that $A \cup B$ is absolutely nonmeasurable with respect to the class of all $\Pi_2$-invariant extensions of the Lebesgue measure $\lambda_2$.

A.B. Kharazishvili *Questions in the theory of sets and in measure theory*, TSU, Tbilisi, 1978
Let $\mathcal{M}(\mathbb{R}^2)$ be a class of all nonzero $\sigma$-finite translation invariant measures on $\mathbb{R}^2$.

**Definition**

A set $X \subset \mathbb{R}^2$ is called *negligible* with respect to $\mathcal{M}(\mathbb{R}^2)$ if these two conditions are satisfied for $X$:

- there exists a measure $\nu \in \mathcal{M}(\mathbb{R}^2)$ such that $X \in \text{dom}(\nu)$;
- for any measure $\mu \in \mathcal{M}(\mathbb{R}^2)$, the relation $X \in \text{dom}(\mu)$ implies the equality $\mu(X) = 0$.

A proper subclass of negligible sets, consisting of the so called absolutely negligible sets, is of special interest for the general theory of invariant measures.
A set $X \subset \mathbb{R}^2$ is called \textit{absolutely negligible} with respect to $M(\mathbb{R}^2)$ if, for every measure $\mu \in M(\mathbb{R}^2)$, there exists a measure $\mu' \in M(\mathbb{R}^2)$ such that the relations

$$\mu' \text{ extends } \mu, \ Y \in \text{dom}(\mu'), \mu'(Y) = 0$$

hold true.
In the paper
has proved next statement:

**Lemma**

If $X \subset \mathbb{R}^2$ is finite in some direction $\vec{e}$, then $M$ is negligible with respect to the class $M(\mathbb{R}^2)$. 
Every Hamel basis of the space $\mathbb{R}^n$ is absolutely negligible subset of $\mathbb{R}^n$.

Notice that a more general result can be stated. For any natural number $n$, denote by $H_n$ the set of all those vectors in $\mathbb{R}^2$ whose representation via the Hamel basis $H$ contains at most $n$ nonzero rational coefficients. Then each set $H_n, n < \omega$ turns out to be $\mathbb{R}^2$-absolutely negligible in $\mathbb{R}^2$.

Theorem

There exists a uniform subset of $\mathbb{R}^2$ which is Hamel basis of $\mathbb{R}^2$.

Remark: The proof of this result is similar to the proof of the fact that there exists a Mazurkiewicz set in $\mathbb{R}^2$ which is a Hamel basis of $\mathbb{R}^2$. 
In general, the solution of the Luzin Problem by Davis and the character of the uniform set infer that any uniform subset of $\mathbb{R}^2$ is $\Pi_2$-negligible and not $\Pi_2$-absolutely negligible.

In connection with this fact is interesting next question
Does there exist a subset of the Euclidean space $\mathbb{R}^n$ which is $\Pi_n$-absolutely negligible and simultaneously, $D_n$-absolutely nonmeasurable? Where, $D_n$ is the group of all motions (i.e. isometric transformations) of $\mathbb{R}^n$ and $\Pi_n$ the group of all translations of the space $\mathbb{R}^n$. 
References


