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The Uniform Subsets of the Eucliedean Plane

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Uniform Set

Definition

Let X be a subset of $\mathbf{R}^2$ and $\vec{e}$ is an arbitrary vector in $\mathbf{R}^2$, X called an Uniform subset of $\mathbf{R}^2$ in direction $\vec{e}$ if for each p' parallel to $\vec{e}$, we have

$$card(p' \cap X) \leq 1$$

.

According to the standard terminology in *N. N. Luzin, Collected Works (in Russian), Izd. Akad. Nauk SSSR, Moscow, 2 (1958)*

Luzini Problem

Many years ago Luzin posed a problem, in particular Luzin asked whether there exists a function

$$\phi : \mathbf{R} \to \mathbf{R}$$

such that the whole plane $\mathbf{R}^2$ can be covered by countable many isometric copies of the graph of ϕ.

Sierpinski Partition

Partially, Sierpinski has answered to the Luzini Problem and under the Continuum Hypothesis has proved next theorem.

Sierpinski's Theorem.

Assuming Continuum Hypothesis in $\mathbf{R^2}$ there exists two subsets A and B, such that

1. The set A is uniform with respect to the axis $\mathbf{R} \times 0$;
2. The set B is uniform with respect to the axis $0 \times \mathbf{R}$;
3. There exists a countable family $\{h_n : n > \omega\}$ of translations of $\mathbf{R}^2$, for which we have
$$\cup\{h_n(A \cup B) : n < \omega\} = \mathbf{R}^2$$

.

Solution of Luzini Problem without CH

Theorem of Davies

Let $(\vec{e}_i)_{i\in\omega}$ be an injective countable family of vectors in $\mathbf{R^2}$. Then there exists a family $\{X_i : i \in \omega\}$ of subsets of $\mathbf{R}^2$ such that

1. $\cup\{X_i : k \in \omega\} = \mathbf{R}^2$;
2. for each $i \in \omega$ the set X_i is uniform in direction $\vec{e}_i$.

Finite set in direction vector $\vec{e}$

Definition

Let $\vec{e}$ be an arbitrary nonzero vector in $\mathbf{R^2}$. A set $B \subset \mathbf{R}^2$ is finite in direction $\vec{e}$ if

$$card(l \cap B) < \omega$$

for any straight line $l \subset \mathbf{R^2}$ parallel to $\vec{e}$.

Theorem

Let $Z \subset \mathbf{R}^2$ be a finite set in direction $\vec{e}$, where $\vec{e}$ is an arbitrary vector in the plane, then there exists an uniform set $X \subset \mathbf{R^2}$ in the same $\vec{e}$ direction, such that Z is a countable many Π_2-configuration of X.

Let E be a set and let M be a class of measures on E (in general, we do not assume that measures belonging to M are defined on the one and same σ-algebra of subset of E).

Definition

- We say that a set $X \subset E$ is absolutely measurable with respect to M if X is measurable with respect to all measures from M.
- We say that a set $Y \subset E$ is relatively measurable with respect to M if there exists at least one measure μ from M such that Y is μ-measurable.
- We say that a set $Z \subset E$is absolutely nonmeasurable with respect to M if there exists no measure from M such that Z is measurable with respect to all measures from M.

Measurability of the Uniform subset

Let Π_2 denote the group of all translations of the plane $\mathbf{R}^2$ and let λ_2 stand for the ordinary two-dimensional Lebesgue measure on $\mathbf{R}^2$.

Theorem

There exists a Π_2-invariant extension μ of the Lebesgue measure λ_2, such that all uniform sets in direction Oy-axis are measurable with respect μ.

Corollary. The uniform set in any direction in $\mathbf{R}^2$ is absolutely measurable with respect to the class of all nonzero σ-finite Π_2-invariant measures.

Theorem

Under **CH**, there exist a set A uniform in direction of Oy-axis and a set B uniform in direction of Ox-axis, such that $A \cup B$ is absolutely nonmeasurable with respect to the class of all Π_2-invariant extensions of the Lebesgue measure λ_2.

A.B. Kharazishvili *Questions in the theory of sets and in measure theory* , TSU, Tblisi, 1978

Auxiliary notions

Let $M(\mathbf{R}^2)$ be a class of all nonzero σ-finite translation invariant measures on $\mathbf{R}^2$.

Definition

A set $X \subset \mathbf{R}^2$ is called *negligible* with respect to $M(\mathbf{R}^2)$ if these two conditions are satisfied for X:

- there exists a measure $\nu \in M(\mathbf{R}^2)$ such that $X \in dom(\nu)$;
- for any measure $\mu \in M(\mathbf{R}^2)$, the relation $X \in dom(\mu)$ implies the equality $\mu(X) = 0$

A proper subclass of negligible sets, consisting of the so called absolutely negligible sets, is of special interest for the general theory of invariant measures.

Auxiliary notions

Definition

A set $X \subset \mathbf{R}^2$ is called *absolutely negligible* with respect to $M(\mathbf{R}^2)$ if, for every measure $\mu \in M(\mathbf{R}^2)$, there exists a measure $\mu' \in M(\mathbf{R}^2)$ such that the relations

$$\mu' \quad \text{extends} \quad \mu, Y \in dom(\mu'), \mu'(Y) = 0$$

hold true.

In the paper

A. Kharazishvili, *Small sets in uncountable abelian groups*, Acta Univ. Lodz, Folia, Math No. 7 (1995) 31-39

has proved next statement:

Lemma

If $X \subset \mathbf{R}^2$ is finite in some direction $\vec{e}$, then M is negligible with respect to the class $M(\mathbf{R}^2)$.

Auxiliary proposition

Lemma

Every Hamel basis of the space $\mathbf{R}^n$ is absolutely negligible subset of $\mathbf{R}^n$.

Notice that a more general result can be stated. For any natural number n, denote by H_n the set of all those vectors in $\mathbf{R}^2$ whose representation via the Hamel basis H contains at most n nonzero rational coefficients. Then each set $H_n, n < \omega$ turns out to be $\mathbf{R}^2$-absolutely negligible in $\mathbf{R}^2$.

[A. Kharazishvili, *One property of Hamel bases*, Bull. Acad. Sci. GSSR, 95, 2 (1979), 277-280.]

Theorem

There exists a uniform subset of $\mathbf{R}^2$ which is Hamel basis of $\mathbf{R}^2$.

Remark: The proof of this result is similar to the proof of the fact that there exists a Mazurkiewicz set in $\mathbf{R}^2$ which is a Hamel basis of $\mathbf{R}^2$.

Conclusion

In general, the solution of the Luzin Problem by Davis and the character of the uniform set infer that any uniform subset of $\mathbf{R^2}$ is Π_2-negligible and not Π_2-absolutely negligible.
In connection with this fact is interesting next question

Open Problem

Does there exists a subset of the Euclidean space $\mathbf{R^n}$ which is Π_n-absolutely negligible and simultaneously, D_n-absolutely nonmeasurable? Where, D_n is the group of all motions (i.e. isometric transformations) of $\mathbf{R^n}$ and Π_n the group of all translations of the space $\mathbf{R^n}$

References

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