On the Existence of One-point Time on an Oriented Set

Grushka Ya.I.

Abstract of the Conference WS2020: Prague-Hejnice, Jan 25 – Feb 1, 2020

Institute of Mathematics NAS of Ukraine. 3, Tereschenkivska st. Kyiv. 01024 Ukraine.
***e-mail**: grushka@imath.kiev.ua*

Definition 1 *The ordered pair* $\mathcal{M} = \left(\mathfrak{Bs}(\mathcal{M}), \underset{\mathcal{M}}{\leftarrow}\right)$ *is called an* ***oriented set*** *if and only if* $\mathfrak{Bs}(\mathcal{M})$ *is some non-empty set (*$\mathfrak{Bs}(\mathcal{M}) \neq \emptyset$*) and* $\underset{\mathcal{M}}{\leftarrow}$ *is arbitrary reflexive binary relation on* $\mathfrak{Bs}(\mathcal{M})$*. In this case the set* $\mathfrak{Bs}(\mathcal{M})$ *is named the* ***basic*** *set or the set of all* ***elementary states*** *of the oriented set* $\mathcal{M}$ *and the relation* $\underset{\mathcal{M}}{\leftarrow}$ *is named by the* ***directing relation of changes (transformations)*** *of* $\mathcal{M}$*.*

In the case where the oriented set $\mathcal{M}$ is known in advance, the char $\mathcal{M}$ in the notation $\underset{\mathcal{M}}{\leftarrow}$ will be released, and we will use the notation $\leftarrow$ instead. From an intuitive point of view, oriented sets may be interpreted as the most primitive models of sets of evolving objects.

Definition 2 *Let* $\mathcal{M}$ *be an oriented set and* $\mathbb{T} = (\mathbf{T}, \leq)$ *be a linearly ordered set. A mapping* $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ *is referred to as* ***time*** *on* $\mathcal{M}$ *if the following conditions are satisfied:*

1 *For any elementary state* $x \in \mathfrak{Bs}(\mathcal{M})$ *there exists an element* $t \in \mathbf{T}$ *such that* $x \in \psi(t)$*.*

2 *If* $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$*,* $x_2 \leftarrow x_1$ *and* $x_1 \neq x_2$*, then there exist elements* $t_1, t_2 \in \mathbf{T}$ *such that* $x_1 \in \psi(t_1)$*,* $x_2 \in \psi(t_2)$ *and* $t_1 < t_2$ *(this means that there is a temporal separateness of successive unequal elementary states).*

In this case the elements $t \in \mathbf{T}$ *are called the* ***moments of time****.*

It turns out that any oriented set $\mathcal{M}$ can be chronologized (that is we can define some time on it). To make sure this we may consider any linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, which contains at least two elements ($\mathbf{card}(\mathbf{T}) \geq 2$) and put, $\psi(t) := \mathfrak{Bs}(\mathcal{M}),\ t \in \mathbf{T}$.

Definition 3 *Let* $\mathcal{M}$ *be an oriented set.*

a *The time* $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ *is called by* ***quasi one-point*** *if for any* $t \in \mathbf{T}$ *the set* $\psi(t)$ *is a singleton.*

b *The time* ψ *is called* ***one-point*** *if the following conditions are satisfied:*

(a) the time ψ *is quasi one-point;*

(b) for every $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ *the conditions* $x_1 \in \psi(t_1)$*,* $x_2 \in \psi(t_2)$ *and* $t_1 \leq t_2$*, assure the correlation* $x_2 \leftarrow x_1$*.*

Example 1 *Let us consider an arbitrary mapping* $f : \mathbb{R} \to \mathbb{R}^d$ *(*$d \in \mathbb{N}$*). This mapping can be interpreted as equation of motion of a single material point in the space* $\mathbb{R}^d$*. The mapping* f *generates the oriented set* $\mathcal{M}_f = \left(\mathfrak{Bs}\left(\mathcal{M}_f\right), \underset{\mathcal{M}_f}{\leftarrow}\right)$*, where* $\mathfrak{Bs}\left(\mathcal{M}_f\right) = \mathfrak{R}(f) = \{f(t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^d$ *and for* $x, y \in \mathfrak{Bs}(\mathcal{M})$ *the correlation* $y \underset{\mathcal{M}_f}{\leftarrow} x$ *holds if and only if there exist* $t_1, t_2 \in \mathbb{R}$ *such, that* $x = f(t_1)$*,* $y = f(t_2)$ *and* $t_1 \leq t_2$*. It is easy to verify, that the following mapping is an one-point time on* $\mathcal{M}_f$*:*

$$\psi(t) = \{f(t)\} \subseteq \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathbb{R}.$$

Example 1 makes clear the notion of one-point time. It is evident, that any one-point time is quasi one-point. There exist the counterexamples, which show that the inverse statement, in general, is not true.

Theorem 1 (see [1, 2]) *Any oriented set $\mathcal{M}$ can be quasi one-point chronologized (this means that we can define some quasi one-point time on $\mathcal{M}$).*

On any oriented set $\mathcal{M}$ we introduce the following additional binary relation:

a For every $x, y \in \mathfrak{Bs}(\mathcal{M})$ we note $y \overset{+}{\underset{\mathcal{M}}{\leftarrow}} x$ if and only if $y \underset{\mathcal{M}}{\leftarrow} x$ and $x \underset{\mathcal{M}}{\not\leftarrow} y$.

b In the cases where it does not lead to misunderstanding we use the notation $y \overset{+}{\leftarrow} x$ instead of the record $y \overset{+}{\underset{\mathcal{M}}{\leftarrow}} x$.

Definition 4 *The oriented set $\mathcal{M}$ is called* ***quasi-chain*** *if and only if the following conditions are satisfied:*

- *For any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ it holds at least one from the correlations $x_2 \leftarrow x_1$ or $x_1 \leftarrow x_2$.*
- *For every $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ the conditions $x_3 \overset{+}{\leftarrow} x_2$, $x_2 \leftarrow x_1$ and $x_1 \overset{+}{\leftarrow} x_0$ lead to the correlation $x_3 \overset{+}{\leftarrow} x_0$ (quasitransitivity).*

It is easy to prove that the transitivity of the binary relation $\leftarrow$ on the oriented set $\mathcal{M}$ implies its quasitransitivity. It can be proven that the inverse statement in general is not valid. That is there exist the oriented set $\mathcal{M}$ such that the relation $\underset{\mathcal{M}}{\leftarrow}$ is quasitransitive but not transitive.

The main result of the talk is the following theorem, which gives the necessary and sufficient condition of existence for one-point time on the oriented set.

Theorem 2 (ZF+AC) *The oriented $\mathcal{M}$ set can be one-point chronologized if and only if it is a quasi-chain.*

We emphasize that proof of "only if" part for Theorem 2 does not require the axiom of choice (AC). This axiom is needed only for the proof of "if" part of Theorem 2. Note that Theorem 2 have been announced in [3].

Using Theorem 2, we can solve the problem of describing all possible images of linearly ordered sets. This problem naturally arises in the theory of ordered sets.

Let $\mathcal{M}$ be an oriented set and $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ be any mapping from $\mathfrak{Bs}(\mathcal{M})$ to $\mathcal{X}$. An oriented set $\mathcal{M}_1$ is referred to as ***image*** of the oriented set $\mathcal{M}$ under the mapping $\mathbf{U}$ if and only if:

1. $\mathfrak{Bs}(\mathcal{M}_1) = \mathbf{U}[\mathfrak{Bs}(\mathcal{M})] = \{\mathbf{U}(x) \mid x \in \mathfrak{Bs}(\mathcal{M})\}$.
2. For $\widetilde{x}, \widetilde{y} \in \mathfrak{Bs}(\mathcal{M}_1)$ the correlation $\widetilde{y} \underset{\mathcal{M}_1}{\leftarrow} \widetilde{x}$ holds if and only if there exist $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $\widetilde{x} = \mathbf{U}(x)$, $\widetilde{y} = \mathbf{U}(y)$ and $y \underset{\mathcal{M}}{\leftarrow} x$.

If is easy to verify that for each mapping $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ there exists an unique image of $\mathcal{M}$ under the mapping $\mathbf{U}$. We will use the notation $\mathbf{U}[[\mathcal{M}]]$ for this image.

It is evidently that every linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ is an oriented set with:

$$\mathfrak{Bs}(\mathbb{T}) = \mathbf{T}, \qquad \underset{\mathbb{T}}{\leftarrow} = \leq .$$

Therefore, it is meaningful to consider the image of the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ under some mapping of kind $\mathbf{U} : \mathbf{T} \to \mathcal{X}$. And the image of the linearly ordered set $\mathbb{T}$ is the oriented set $\mathbf{U}[[\mathbb{T}]]$. That is why the following problem naturally arises:

Problem 1 *Can an arbitrary oriented set be represented as the image* $\mathbf{U}[[\mathbb{T}]]$ *of some linearly ordered set* $\mathbb{T}$*? In the case of negative answer to the previous question, it is interesting to describe all oriented sets that can be represented as an image of some linearly ordered set.*

Applying Theorem 2, we are able to give the solution of Problem 1:

Corollary 1 *An oriented set* $\mathcal{M}$ *can be represented as image of some linearly ordered set if and only if it is a quasi-chain.*

References

[1] Grushka Ya.I. *Primitive changeable sets and their properties.* Mathematical Bulletin of the Shevchenko Scientific Society, **9** (2012), 52-80, (in Ukrainian).

[2] Grushka Ya.I. *Draft introduction to abstract kinematics. (Version 2.0).* Preprint: ResearchGate, (2017), DOI: https://doi.org/10.13140/RG.2.2.28964.27521.

[3] Grushka Ya.I. *Necessary and sufficient condition for the existence of the one-point time on an oriented set.* Reports of the National Academy of Sciences of Ukraine, **2019** (8), (2019), 9–15, DOI: https://doi.org/10.15407/dopovidi2019.08.009, (in Ukrainian).