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On the Existence of One-point Time on an Oriented Set

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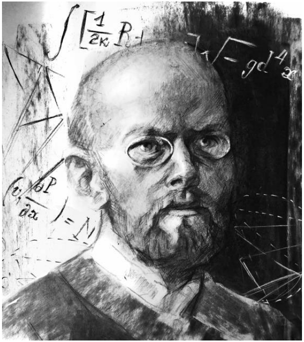
1. Motivation. On changeable sets and sixth Hilbert problem

A few Words about the Sixth Hilbert Problem

The notion of oriented set is the basic elementary concept of the theory of changeable sets.



The main motivation for building the theory of changeable sets is generated by the sixth Hilbert problem, that is, the problem of mathematically rigorous formulation of the fundamentals of theoretical physics. This problem was posed in 1900, but it remains very relevant today. In particular in may 2016 an international workshop dedicated to this problem was held in Leicester (England). For more details see the paper [1].

[1] A.N. Gorban. *Hilbert's sixth problem: the endless road to rigour.* // Phil. Trans. R. Soc. A. – 376: 20170238. – 2018. – http://dx.doi.org/10.1098/rsta.2017.0238.

Changeable Sets from Intuitive Point of View

From an intuitive point of view changeable sets are the sets of objects, which (unlike elements of ordinary (static) sets) can be in process of continuous transformations. In particular, these objects can change their properties, appear or disappear, break down into several parts or, conversely, unite into a single unit. Moreover, the evolution picture of a changeable set may depend of the area of observation or reference frame.

Example 1 (an informal example of a changeable set)

The set of all sparrows of Czech Republic is a set in the sense of classical set theory only in the case, where we observe it in some fixed time point. But if we observe this set during some time interval, we must emphasize that this set does not consist of the changeless composition of elements, moreover its elements may change their properties.

The problem of mathematical theory of changeable sets

The problem of constructing the mathematical theory of changeable sets (that is the "sets" possessing the properties listed above) was emerged in the papers of Russian biophysicist **Alexander Levich**, as well as in the papers of Michael Barr, John L. Bell, etc (see [2–4]).

[2] Levich A.P. *Modeling of "dynamic sets"*. // in collection "Irreversible processes in nature and technique" (by MSTU named after N.E.~Bauman). – P. 3–46.

[3] Michael Barr, Colin Mclarty, Charles Wells. *Variable Set Theory*. – 1986. – 12 p. – http://www.math.mcgill.ca/barr/papers/vst.pdf.

[4] John L. Bell. *Abstract and Variable Sets in Category Theory*. // In the book "What is Category Theory?". – Monza (Italy): Polimetrica International Scientific Publisher. – 2006. – P. 9-16.

In its present form, the theory of changeable sets was constructed in [5-9] etc.

[5] Grushka Ya.I. Changeable sets and their properties. // Reports of the National Academy of Sciences of Ukraine. – 2012. – No 5. – P. 12-18. (in Ukrainian) – https://www.researchgate.net/publication/236120448.

[6] Grushka Ya.I. Primitive changeable sets and their properties // Mathematical Bulletin of the Shevchenko Scientific Society. – 2012. – V. **9**. – P. 52-80. – https://www.researchgate.net/publication/236120647.

[7] Grushka Ya.I. Base Changeable Sets and Mathematical Simulation of the Evolution of Systems. // Ukrainian Mathematical Journal. – 2013. – V. **65**, No 9. – P. 1198-1218. – http://doi.org/10.1007/s11253-014-0862-6.

[8] Grushka Ya.I. Abstract concept of changeable set. – Preprint: arXiv:1207.3751, 2012. – P. 1-54.

[9] Grushka Ya.I. Draft introduction to abstract kinematics. (Version 2.0). – Preprint: ResearchGate, 2017. – P. 1-208. – https://doi.org/10.13140/RG.2.2.28964.27521.

Scheme of Hierarchy of Mathematical Objects of the Theory of Changeable Sets

Oriented sets

↓

Primitive changeable sets

↓

Base changeable sets

↓

Changeable sets

▶ This talk focuses on the first "step" of the hierarchy of mathematical objects of the theory of changeable sets. Namely, we consider **oriented sets** as well **theorems of** their **chronologization**.

2. On Oriented Sets and Chains

Definition of Oriented Set

Oriented sets may be interpreted as simplest abstract models of sets of changing objects, evolving in the framework of the single (specified) reference frame.

Definition 1

The ordered pair $\mathcal{M} = \left(\mathfrak{Bs}(\mathcal{M}), \underset{\mathcal{M}}{\leftarrow}\right)$ is called an **oriented set** if and only if $\mathfrak{Bs}(\mathcal{M})$ is some non-empty set ($\mathfrak{Bs}(\mathcal{M}) \neq \emptyset$) and $\underset{\mathcal{M}}{\leftarrow}$ is arbitrary reflexive binary relation on $\mathfrak{Bs}(\mathcal{M})$. In this case the set $\mathfrak{Bs}(\mathcal{M})$ is named the **basic** set or the set of all **elementary states** of the oriented set $\mathcal{M}$ and the relation $\underset{\mathcal{M}}{\leftarrow}$ is named by the **directing relation of changes (transformations)** of $\mathcal{M}$.

In the case where the oriented set $\mathcal{M}$ is known in advance we use the abbreviated notation $\leftarrow$ instead of $\underset{\mathcal{M}}{\leftarrow}$.

For elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ the record $y \leftarrow x$ should be understood as "the elementary state y is the result of transformations (or the transformation offspring) of the elementary state x".

Transitive Subsets, Chains

It is clear that partially-ordered and quasi-ordered sets are particular cases of oriented sets.

Definition 2

Let $\mathcal{M}$ be an oriented set.

- The non-empty subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ is referred to as **transitive** in $\mathcal{M}$ if for any $x, y, z \in N$ the correlations $z \leftarrow y$ and $y \leftarrow x$ lead to $z \leftarrow x$.
- The transitive subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ is called by **maximum transitive** if there does not exist a transitive subset $N_1 \subseteq \mathfrak{Bs}(\mathcal{M})$, such, that $N \subset N_1$ (where the symbol $\subset$ denotes the strict inclusion, that is $N \neq N_1$).
- The transitive subset $L \subseteq \mathfrak{Bs}(\mathcal{M})$ is referred to as **chain** in $\mathcal{M}$ if for any $x, y \in L$ at least one of the correlations $y \leftarrow x$ or $x \leftarrow y$ is true. The chain $L \subseteq \mathfrak{Bs}(\mathcal{M})$ we will name by the **maximum chain** if there does not exist a chain $L_1 \subseteq \mathfrak{Bs}(\mathcal{M})$, such, that $L \subset L_1$.

The next two statements follow from the axiom of choice.

Assertion 1

For any transitive set N of oriented set $\mathcal{M}$ there exists a maximum transitive set $N_{\max}$ such, that $N \subseteq N_{\max}$.

Assertion 2

For any chain L of oriented set $\mathcal{M}$ there exists a maximum chain $L_{\max}$ such, that $L \subseteq L_{\max}$.

- Assertion 2 can be interpreted to as the generalization of the Hausdorff maximal principle. So this assertion is logical equivalent to the axiom of choice
- **Question:** Is Assertion 1 logical equivalent to the axiom of choice ?

3. Time on Oriented Sets. Some Chronologization Theorems

Remark on Time Scales

In theoretical physics, scientists tend to think, that the moments of time are real numbers. But, despite this, in the next definition we will not be restricted to the moments of time belonging to the set of real numbers $\mathbb{R}$, namely in this definition the moments of time are elements of any linearly ordered set. Such definition of time is close to the philosophical conception of time as some "chronological order", somehow agreed with the processes of evolution.

Recall that linearly ordered set is an ordered pair of kind $\mathbb{T} = (\mathbf{T}, \leq)$ with reflexive, asymmetric and transitive binary relation $\leq$ on $\mathbf{T}$, satisfying the following additional condition:

(LnO) for every $t, \tau \in \mathbf{T}$ it is performed at least one of the correlations $t \leq \tau$ or $\tau \leq t$.

Definition of Time on Oriented Sets

Definition 3

Let $\mathcal{M}$ be an oriented set and $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set. A mapping $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ is referred to as **time** on $\mathcal{M}$ if the following conditions are satisfied:

1. For any elementary state $x \in \mathfrak{Bs}(\mathcal{M})$ there exists an element $t \in \mathbf{T}$ such that $x \in \psi(t)$.
2. If $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$, $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, then there exist elements $t_1, t_2 \in \mathbf{T}$ such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$ (this means that there is a temporal separateness of successive unequal elementary states).

In this case the elements $t \in \mathbf{T}$ are called by the **moments of time** and the pair $\mathcal{H} = (\mathbb{T}, \psi) = ((\mathbf{T}, \leq), \psi)$ we name by **chronologization** of $\mathcal{M}$.

Remark on Possibility of Chronologization

We say that an oriented set $\mathcal{M}$ ***can be chronologized*** if there exists at least one chronologization of $\mathcal{M}$.

▶ It turns out that any oriented set $\mathcal{M}$ can be chronologized. To make sure this we may consider any linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, which contains at least two elements $(\mathbf{card}(\mathbf{T}) \geq 2)$ and put:

$$\psi(t) := \mathfrak{Bs}(\mathcal{M}), \ t \in \mathbf{T}.$$

Some more non-trivial methods to chronologize an oriented set will be considered below. Namely we will consider the time with some additional properties.

Quasi One-point and One-point Time

Definition 4

Let $\mathcal{M}$ be an oriented set and $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set.

1) The time $\psi : \mathbf{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ will be called **quasi one-point** if for every $t \in \mathbf{T}$ the set $\psi(t)$ is a singleton.

2) The time ψ will be called **one-point** if the following conditions are satisfied:

(a) The time ψ is quasi one-point.

(b) For any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ the conditions $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$, lead to $x_2 \leftarrow x_1$.

Example of One-point Time

Example 2

Let us consider an arbitrary mapping $f : \mathcal{I} \to \mathbb{R}^d$ $(d \in \mathbb{N})$, where $\mathcal{I} \subseteq \mathbb{R}$ is some connected subset of Real axis $\mathbb{R}$. This mapping can be interpreted as equation of motion of single material point in the space $\mathbb{R}^d$. This mapping f generates the oriented set $\mathcal{M}_f = \left(\mathfrak{Bs}\left(\mathcal{M}_f\right), \underset{\mathcal{M}_f}{\leftarrow} \right)$, where $\mathfrak{Bs}\left(\mathcal{M}_f\right) = \mathfrak{R}(f) = \{f(t) \,|\, t \in \mathcal{I}\} \subseteq \mathbb{R}^d$ and for $x, y \in \mathfrak{Bs}\left(\mathcal{M}_f\right)$, the correlation $y \underset{\mathcal{M}_f}{\leftarrow} x$ is valid if and only if there exist $t_1, t_2 \in \mathcal{I}$ such, that $x = f\left(t_1\right)$, $y = f\left(t_2\right)$ and $t_1 \leq t_2$. It is easy to verify, that the following mapping is an one-point time on $\mathcal{M}_f$:

$$\psi_f(t) = \{f(t)\} \subseteq \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathcal{I}.$$

- Example 2 makes more clear the notion of one-point time.
- It is evident, that any one-point time is quasi one-point. There exist examples, which show that the inverse statement is not true in the general case.

Theorem on quasi one-point chronologization

Theorem 1 (ZF+LO, see [6,9])

Any oriented set $\mathcal{M}$ can be quasi one-point chronologized (this means that we can define some quasi one-point time on $\mathcal{M}$).

Remark. Proof of Theorem 1 uses the Linear Ordering principle (LO) in addition to Zermelo–Fraenkel axiomatic system (ZF). This principle asserts that any set can be linearly ordered. It is evident that the above principle follows from the famous well-ordering Zermelo's theorem, and therefore, from the axiom of choice (AC). But it is known that LO-principle also follows from Ultrafilter Theorem of Tarski and, moreover, it is logically weaker than this theorem and therefore than the axiom of choice [10, p. 17, 18].

[10] Horst Herrlich. Axiom of Choice. – Springer - 2006. – DOI: 10.1007/11601562.

Theorem on one-point chronologization

- We name an oriented set $\mathcal{M}$ by a **chain oriented set** if the set $\mathfrak{Bs}(\mathcal{M})$ is the chain of $\mathcal{M}$, that is if the relation $\leftarrow$ if transitive on $\mathfrak{Bs}(\mathcal{M})$ and for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ at least one of the conditions $x \leftarrow y$ or $y \leftarrow x$ is satisfied (note that in this case the oriented set $\mathcal{M}$ is a linearly quasi-ordered set).

Theorem 2 (see [6,9])

Any chain oriented set can be one-point chronologized.

It turns out that Theorem 2 is not reversible. And the next example demonstrates the existence of non-chain oriented sets, which can be one-point chronologized (see the next frame).

Example 3

Consider the function $f_0 : [0, 2\pi] \to \mathbb{R}^2$, defined by the formula:

$$f_0(t) = (\cos t, \sin t) \qquad (t \in [0, 2\pi]) .$$

According to Example 2, using this function, we may construct the oriented set $\mathcal{M}_{f_0}$. This oriented set can be one-point chronologized by mens of the time:

$$\psi_{f_0}(t) = \{f_0(t)\} \qquad (t \in [0, 2\pi]).$$

At the same time, this oriented set is not a chain, because it is easy to verify that the binary relation $\xleftarrow[\mathcal{M}_{f_0}]{}$ is not transitive on $\mathfrak{Bs}(\mathcal{M}_{f_0})$.

Problem 1

Find necessary and sufficient conditions of existence of one-point chronologization for oriented set.

The Main Theorem

- For any elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ of an arbitrary oriented set we use the notation $y \overset{+}{\leftarrow} x$ if and only if $y \leftarrow x$ and $x \not\leftarrow y$.
- The oriented set $\mathcal{M}$ is called a **quasi-chain** if and only if the following conditions are satisfied:

1. For any $x, y \in \mathfrak{Bs}(\mathcal{M})$ it holds at least one from the correlations $y \leftarrow x$ or $x \leftarrow y$;
2. $\forall x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ the conditions $x_3 \overset{+}{\leftarrow} x_2$, $x_2 \leftarrow x_1$ and $x_1 \overset{+}{\leftarrow} x_0$ lead to the correlation $x_3 \overset{+}{\leftarrow} x_0$ (quasitransitivity).

It is easy to prove that every chain oriented set is a quasi-chain. Using Example 3 it can be proven that the inverse statement in general is not valid.

Theorem 3 (ZF+AC)

The oriented set $\mathcal{M}$ can be one-point chronologized if and only if it is a quasi-chain.

The Problem of Image of Linearly Ordered Set

Using Theorem 3, we can solve the problem of describing all possible images of linearly ordered sets. This problem naturally arises in the theory of ordered sets.

Definition 5

Let $\mathcal{M}$ be an oriented set and $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ be any mapping from $\mathfrak{Bs}(\mathcal{M})$ to $\mathcal{X}$. An oriented set $\mathcal{M}_1$ is referred to as ***image*** of the oriented set $\mathcal{M}$ under the mapping $\mathbf{U}$ if and only if:

1. $\mathfrak{Bs}\left(\mathcal{M}_1\right) = \mathbf{U}\left[\mathfrak{Bs}(\mathcal{M})\right] = \{\mathbf{U}(x) \mid x \in \mathfrak{Bs}(\mathcal{M})\}$.
2. For $\widetilde{x}, \widetilde{y} \in \mathfrak{Bs}\left(\mathcal{M}_1\right)$ the correlation $\widetilde{y} \underset{\mathcal{M}_1}{\leftarrow} \widetilde{x}$ holds if and only if there exist $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $\widetilde{x} = \mathbf{U}(x)$, $\widetilde{y} = \mathbf{U}(y)$ and $y \underset{\mathcal{M}}{\leftarrow} x$.

If is easy to verify that for each mapping $\mathbf{U} : \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ there exists an unique image of $\mathcal{M}$ under the mapping $\mathbf{U}$. We will use the notation $\mathbf{U}\left[\left[\mathcal{M}\right]\right]$ for this image.

It is evidently that every linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ is an oriented set, where:

$$\mathfrak{Bs}(\mathbb{T}) = \mathbf{T}, \qquad \underset{\mathbb{T}}{\leftarrow} = \leq .$$

Therefore, it is meaningful to consider the image of the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ under some mapping $\mathbf{U} : \mathbf{T} \to \mathcal{X}$. And the image of the linearly ordered set $\mathbb{T}$ is the oriented set $\mathbf{U}[[\mathbb{T}]]$. So, we have the following problem:

Problem 2

Can an arbitrary oriented set be represented as the image $\mathbf{U}[[\mathbb{T}]]$ of some linearly ordered set $\mathbb{T}$? In the case of negative answer to the previous question, it is interesting to describe all oriented sets that can be represented as an image of some linearly ordered set.

Corollary 1 (of Theorem 3)

An oriented set $\mathcal{M}$ can be represented as image of some linearly ordered set if and only if it is a quasi-chain.

Thank you for attention!