Topological applications of Wadge theory II

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Reasonably closed Wadge classes

Given $i \in 2$, set:

$$Q_i = \{ x \in 2^\omega : x(n) = i \text{ for all but finitely many } n \in \omega \}$$

Notice that every element of $2^\omega \setminus (Q_0 \cup Q_1)$ is obtained by alternating finite blocks of zeros and finite blocks of ones.

Define the function $\phi : 2^\omega \setminus (Q_0 \cup Q_1) \rightarrow 2^\omega$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if the } n^{\text{th}} \text{ block of zeros of } x \text{ has even length} \\ 1 & \text{otherwise} \end{cases}$$

where we start counting with the $0^{\text{th}}$ block of zeros. It is easy to check that $\phi$ is continuous.

**Definition (Steel, 1980)**

Let $\Gamma$ be a Wadge class in $2^\omega$. We will say that $\Gamma$ is *reasonably closed* if $\phi^{-1}[A] \cup Q_0 \in \Gamma$ for every $A \in \Gamma$. 
Why would anybody need that?

**Lemma (Harrington)**

Let $\Gamma = [B]$ be a reasonably closed Wadge class in $2^\omega$. If $A \leq B$ then this is witnessed by an injective function.

The above lemma will be useful to us because every injective continuous function $f : 2^\omega \rightarrow 2^\omega$ is an embedding.

**Proof.**

Let $A^* = \phi^{-1}[A] \cup Q_0$. Since $\Gamma$ is reasonably closed, we can fix $\sigma : 2^{<\omega} \rightarrow 2^{<\omega}$ such that $f_\sigma : 2^\omega \rightarrow 2^\omega$ witnesses $A^* \leq B$. We will construct $\tau : 2^{<\omega} \rightarrow 2^{<\omega}$ such that $f_\tau : 2^\omega \rightarrow 2^\omega$ witnesses $A \leq A^*$ and $f_\sigma \circ f_\tau$ is injective.

Make sure that

1. $\tau(s)$ always ends with a 1
2. There are exactly $|s|$ blocks of zeros in $\tau(s)$
3. $s(n)$ is the parity of the $n^{th}$ block of zeros in $\tau(s)$
Begin by setting $\tau(\emptyset) = \langle 1 \rangle$.

Given $s \in 2^{<\omega}$, notice that $\tau(s) \vec{0} \in A^*$ and $\tau(s) \vec{1} \notin A^*$.

Since $f_\sigma$ witnesses that $A^* \leq B$, we must have $f_\sigma(\tau(s) \vec{0}) \in B$ and $f_\sigma(\tau(s) \vec{1}) \notin B$. Therefore, we can find $k \in \omega$ such that

$$\sigma(\tau(s) \vec{0}^k) \neq \sigma(\tau(s) \vec{1}^k)$$

Now simply pick $\tau(s \vec{i}) \supseteq \tau(s \vec{i}^k)$ for $i = 0, 1$ satisfying conditions (1), (2) and (3).

To check that $f_\tau$ has the desired properties, observe that

- $\text{ran}(f_\tau) \subseteq 2^\omega \setminus (Q_0 \cup Q_1)$ (By conditions 1 and 2)

- $\phi(f_\tau(x)) = x$ for every $x \in 2^\omega$ (By conditions 1 and 3)
Our main tool: Steel’s theorem

Given a Wadge class $\Gamma$ in $2^\omega$ and $X \subseteq 2^\omega$, we will say that $X$ is everywhere properly $\Gamma$ if $X \cap [s] \in \Gamma \setminus \tilde{\Gamma}$ for every $s \in 2^{<\omega}$.

Theorem (Steel, 1980)

Let $\Gamma$ be a reasonably closed Wadge class in $2^\omega$. Assume that $X$ and $Y$ are subsets of $2^\omega$ that satisfy the following:

- $X$ and $Y$ are everywhere properly $\Gamma$
- $X$ and $Y$ are either both meager or both comeager

Then there exists a homeomorphism $h : 2^\omega \rightarrow 2^\omega$ such that $h[X] = Y$.

Proof.

Without loss of generality, fix closed nowhere dense subsets $X_n$ and $Y_n$ of $2^\omega$ for $n \in \omega$ such that $X \subseteq \bigcup_{n \in \omega} X_n$ and $Y \subseteq \bigcup_{n \in \omega} Y_n$. We will combine Harrington’s Lemma with Knaster-Reichbach systems. (To be continued...)
Knaster-Reichbach covers
Fix a homeomorphism $h : C \to D$ between closed nowhere dense subsets of $2^\omega$. We will say that $\langle U, V, \psi \rangle$ is a Knaster-Reichbach cover (briefly, a KR-cover) for $\langle 2^\omega \setminus C, 2^\omega \setminus D, h \rangle$ if the following conditions hold:

- $U$ is a cover of $2^\omega \setminus C$ consisting of pairwise disjoint non-empty clopen subsets of $2^\omega$
- $V$ is a cover of $2^\omega \setminus D$ consisting of pairwise disjoint non-empty clopen subsets of $2^\omega$
- $\psi : U \to V$ is a bijection
- If $f : 2^\omega \to 2^\omega$ is a bijection such that $h \subseteq f$ and $f[U] = \psi(U)$ for every $U \in U$ (we say that $f$ respects $\psi$), then $f$ is continuous on $C$ and $f^{-1}$ is continuous on $D$

Lemma (see Medini, 2015)
Let $h : C \to D$ be a homeomorphism between closed nowhere dense subsets of $2^\omega$. Then there exists a KR-cover for $\langle 2^\omega \setminus C, 2^\omega \setminus D, h \rangle$. 
Knaster-Reichbach covers
Knaster-Reichbach covers
Knaster-Reichbach covers

\[ \mathcal{C} \xrightarrow{h} \mathcal{D} \]
Knaster-Reichbach covers
Knaster-Reichbach systems

Fix an admissible metric on $2^\omega$. We will say that a sequence $\langle \langle h_n, K_n \rangle : n \in \omega \rangle$ is a *Knaster-Reichbach system* (briefly, a KR-system) if the following conditions are satisfied:

- Each $h_n : C_n \to D_n$ is a homeomorphism between closed nowhere dense subsets of $2^\omega$
- $h_m \subseteq h_n$ whenever $m \leq n$
- Each $K_n = \langle U_n, V_n, \psi_n \rangle$ is a KR-cover for $\langle 2^\omega \setminus C_n, 2^\omega \setminus D_n, h_n \rangle$
- $\text{mesh}(U_n) \leq 2^{-n}$ and $\text{mesh}(V_n) \leq 2^{-n}$ for each $n$
- $U_m$ refines $U_n$ and $V_m$ refines $V_n$ whenever $m \geq n$
- Given $U \in U_m$ and $V \in U_n$ with $m \geq n$, then $U \subseteq V$ if and only if $\psi_m(U) \subseteq \psi_n(V)$
Knaster-Reichbach systems
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Knaster-Reichbach systems
Why do we care about Knaster-Reichbach systems?

Because they give us homeomorphisms!

Theorem (see Medini, 2015)

Assume that \(\langle h_n, K_n \rangle : n \in \omega \) is a KR-system. Then there exists a homeomorphism \( h : 2^\omega \to 2^\omega \) such that \( h \supseteq \bigcup_{n \in \omega} h_n \).

Corollary

Let \( X \) and \( Y \) be subspaces of \( 2^\omega \). Assume that \( \langle h_n, K_n \rangle : n \in \omega \) is a KR-system satisfying the following additional conditions:

\[
\begin{align*}
\text{\( X \subseteq \bigcup_{n \in \omega} C_n \)} \\
\text{\( Y \subseteq \bigcup_{n \in \omega} D_n \)} \\
\text{\( h_n[X \cap C_n] = Y \cap D_n \) for each \( n \)}
\end{align*}
\]

Then there exists a homeomorphism \( h : 2^\omega \to 2^\omega \) such that \( h \supseteq \bigcup_{n \in \omega} h_n \) and \( h[X] = Y \).
Proof of Steel’s theorem
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\[ X_0 \]

\[ 2^{\omega} \]

\[ f \]

witnessing

\[ X \leq Y \cap [s] \]

\[ 2^{\omega} \]

\[ Y_0 \]

\[ [s] \]
Proof of Steel’s theorem

\[ X_0 \rightarrow [t] \]

\[ f \]

witnessing

\[ X \leq Y \cap [s] \]

\[ 2^\omega \]

\[ 2^\omega \]

\[ Y_0 \]

\[ [s] \]
Proof of Steel’s theorem

The diagram illustrates the proof of Steel’s theorem, involving relations between sets $X_0$, $Y$, $Y_0$, and $X \leq Y \cap [s]$. The diagram shows the following:

- $X_0$ is related to $2^\omega$.
- $Y \leq X \cap [t]$ is witnessed by $g$.
- $X \leq Y \cap [s]$ is witnessed by $f$.
- $Y_0$ is related to $2^\omega$.

The diagram visually represents the hierarchical and comparative relationships among these sets.
Proof of Steel’s theorem

Remember that our strategy is to construct a KR-system $\langle\langle h_n, K_n \rangle : n \in \omega \rangle$. We have seen how to begin:

- $C_0 = X_0 \cup g[Y_0]$
- $D_0 = Y_0 \cup f[X_0]$
- $h_0 = (f \upharpoonright X_0) \cup (g^{-1} \upharpoonright g[Y_0])$

Then obtain a KR-cover $\langle U_0, V_0, \psi_0 \rangle$ for $\langle 2^\omega \setminus C_0, 2^\omega \setminus D_0, h_0 \rangle$.

The next step is like the first one, but with the following changes:

- Instead of working between $2^\omega$ and $2^\omega$, work between $U$ and $\psi_0(U)$, where $U \in U_0$
- Instead of looking at $X_0$ and $Y_0$, look at $X_1 \cap U$ and $Y_1 \cap \psi_0(U)$
- Repeat for every $U \in U_0$, then union up the partial homeomorphisms to get $h_1$

Keep going like this for $\omega$ more steps...
Thank you for your attention

and have a good evening!