Closed hereditary coreflective subcategories in categories of Tychonoff spaces

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- epireflective in **Top**

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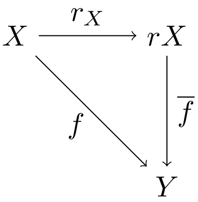
- **A** is reflective in **Top**:
  for each $X \in \mathbf{Top}$ there exists an $rX \in \mathbf{A}$ and a map $r_X : X \to rX$ such that for every $Y \in \mathbf{A}$ and every $f : X \to Y$ there exists a unique $\bar{f} : rX \to Y$ such that the following diagram commutes:



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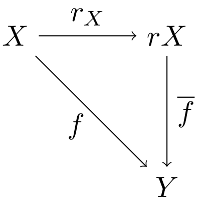

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  $\Leftrightarrow$ closed under the formation of subspaces and products

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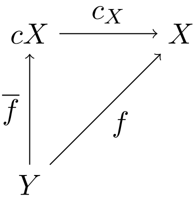

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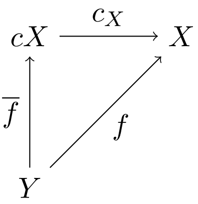

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- closed hereditary coreflective (CHC): closed under the formation of closed subspaces, sums and extremal quotient objects

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If sequential cardinals do not exist, then there are no other CHC subcategories.

- κ is sequential if there exists a sequentially continuous non-continuous map $f : 2^\kappa \to \mathbb{R}$

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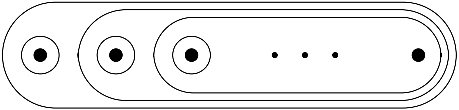

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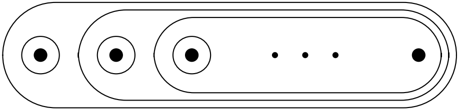

Proposition

The CHC hull of $C(\alpha)$ in $\mathbf{A}$ is $\mathbf{Top}(\alpha) \cap \mathbf{A}$.

Proposition (Sleziak, 2008)

If X is not a sum of connected spaces then there exists a quotient map $f : X \to P$, where P is a prime T_2-space and $P \prec C(\alpha)$.

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Corollary

If $\mathbf{B}$ is CHC in $\mathbf{A}$ and it contains a space that is not a sum of connected spaces, then it contains $C(\alpha)$ for some regular α.

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Let $X = \prod_{i \in I} X_i$ where each X_α is a Tychonoff $s_{\mathbb{R}}$-space. If each X_i is locally pseudocompact, then X is an $s_{\mathbb{R}}$-space if and only if $|I|$ is non-sequential.

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- a space X is a quotient of the sum of its prime factors: $\coprod_{a \in X} X_a \to X$

Thank you for your attention.