Breaking news about Vopěnka’s Principle!

For the purposes of this slide, let $\text{Ord}$ denote partial order category for the (class of all) ordinals with the usual order. Another formulation of Vopěnka’s Principle is

**Vopěnka’s Principle, $\text{Ord}$ embedding formulation**

There is no full embedding of $\text{Ord}$ into the category of graphs.

The dual statement is called **Weak Vopěnka’s Principle**

**Weak Vopěnka’s Principle**

There is no full embedding of $\text{Ord}^{op}$ into the category of graphs.

This is useful for (indeed equivalent to) various statements about accessible categories.
Breaking news about Vopěnka’s Principle!

It has long been known that Vopěnka’s Principle implies Weak Vopěnka’s Principle, which implies there is a proper class of measurable cardinals, but it’s been an open question whether either of these implications could be reversed.

Theorem

"Ord is Woodin" implies Weak Vopěnka’s Principle. In particular, it is strictly weaker than Vopěnka’s Principle.

He also has a claimed proof that Weak Vopěnka’s Principle implies "Ord is Woodin", pinpointing the exact strength of Weak Vopěnka’s Principle. But it’s a much more involved argument that’s still being refereed and which I haven’t read yet; it’s at arXiv:1907.00284.
Breaking news about Vopěnka’s Principle!

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Last year, Trevor Wilson showed:

**Theorem**

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Part III: Strong compactness and an application
Abstract Elementary Classes (AECs)

Notation

In this talk, for a language $L$ and $L$-structures $M$ and $N$, we'll write $M \subseteq L N$ to mean $M$ is an $L$-substructure of $N$, and $|M|$ for the underlying set of the structure $M$.

Definition

A pair $(K, \leq)$ is an abstract elementary class if:

1. $K$ is a class of $L$-structures for a fixed finitary language $L$.
2. $\leq$ is a reflexive and transitive binary relation on $K$.
3. For $M$ and $N$ in $K$, if $M \leq N$ then $M \subseteq L N$.
4. Both $K$ and $\leq$ are closed under isomorphism: if $M, N \in K$, $M \leq N$ and $f : N \sim = N'$, then $f'' M$ and $N'$ are in $K$, and $f'' M \leq N$.

...continued
Abstract Elementary Classes (AECs)

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...continued
Abstract Elementary Classes (AECs)

5 (Coherence axiom) If $M_0, M_1, M_2 \in K$, $M_0 \subseteq_{L} M_1 \leq M_2$, and $M_0 \leq M_2$, then $M_0 \leq M_1$.

6 (Tarki-Vaught chain axioms) If $\delta$ is a limit ordinal and $\langle M_i \mid i < \delta \rangle$ is a $\leq$-increasing chain of members of $K$, then

1. $M_\delta := \bigcup_{i<\delta} M_i \in K$,
2. $M_i \leq M_\delta$ for all $i < \delta$, and
3. if $N \in K$ and $M_i \leq N$ for all $i < \delta$ then $M_\delta \leq N$. 
Abstract Elementary Classes (AECs)

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   3. if $N \in \mathcal{K}$ and $M_i \leq N$ for all $i < \delta$ then $M_\delta \leq N$.

7 (Löwenheim-Skolem-Tarski axiom) There is a cardinal $\mu \geq \|\mathcal{L}(\mathcal{K})\| + \aleph_0$ such that for every $M \leq \mathcal{K}$ and every $A \subseteq |M|$, there exists $M_0 \leq M$ in $\mathcal{K}$ such that $A \subseteq |M|$ and $\|M_0\| \leq \mu + \|A\|$. We define the Löwenheim-Skolem-Tarski number of $\mathcal{K}$, $LS(\mathcal{K})$, to be the least such $\mu$. 
Examples

- The class of all models of a sentence of first order logic, or of any countable fragment of $\mathcal{L}_{\omega_1,\omega}$, with the associated notion of elementary submodel as $\leq$. 
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- Any class of models closed under elementary equivalence, with elementary submodel as $\leq$. Eg: Artinian rings.
- $\omega_1 + 1$ as $\mathbb{K}$, usual ordinal $\leq$ as $\leq$. 
Theorem (Lieberman and Rosický, 2016)

An AEC is precisely an accessible category $\mathcal{K}$ in which every morphism is a monomorphism, colimits exist for all directed diagrams, and there is a faithful functor $U$ to $\textbf{Set}$ satisfying:

- $U$ preserves directed colimits
- coherence: for any commutative diagram

\[
\begin{array}{ccc}
UA & \xrightarrow{U(h)} & UC \\
\downarrow{f} & & \downarrow{U(g)} \\
UB & & \\
\end{array}
\]

there is an $\bar{f}$ in $\mathcal{K}$ such that $U(\bar{f}) = f$.

- there is a canonical language $\mathcal{L}_U$ that can be associated with a functor $U$ as above; if $f : UA \to UB$ is an $\mathcal{L}_U$-structure isomorphism, there is a $\bar{f}$ in $\mathcal{K}$ such that $f = U(\bar{f})$ (iso-fullness).
Doing model theory with AECs: types

The whole flavour of AECs is to avoid syntax, so the usual notion of type isn’t appropriate.
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The whole flavour of AECs is to avoid syntax, so the usual notion of type isn’t appropriate. But if there’s a monster model $\mathcal{M}$ (which is true iff your AEC has joint embedding, amalgamation, and no maximal models) we can define a semantic notion of type, saying that $\bar{b}$ and $\bar{c}$ have the same type over $A$ if there’s an automorphism of $\mathcal{M}$ fixing $A$ and taking $\bar{b}$ to $\bar{c}$.
The whole flavour of AECs is to avoid syntax, so the usual notion of type isn’t appropriate. But if there’s a monster model $\mathcal{M}$ (which is true iff your AEC has joint embedding, amalgamation, and no maximal models) we can define a semantic notion of type, saying that $\bar{b}$ and $\bar{c}$ have the same type over $A$ if there’s an automorphism of $\mathcal{M}$ fixing $A$ and taking $\bar{b}$ to $\bar{c}$. Even without a monster model, we can use this idea to define *Galois types*. 
Part III: Strong compactness and an application

Doing model theory with AECs: Morley’s Theorem

(Grossberg & VanDieren 2006) Any tame AEC with a monster model admits a generalisation of Morley’s categoricity theorem, with a caveat about the starting cardinality: if it only has (up to isomorphism) 1 member of cardinality $\lambda$ for a sufficiently large $\lambda$, then it only has 1 member (up to isomorphism) in each cardinality $\mu \geq \lambda$.

Here, an AEC is said to be $\kappa$-tame if for any two distinct Galois 1-types over some $M \in K$, there is a subset $A$ of $M$ of size less than $\kappa$ over which they are already different.

$K$ is tame if it is $\kappa$-tame for some $\kappa$.

Theorem (Boney 2014) If there is a proper class of strongly compact cardinals, then every AEC is tame.
(Grossberg & VanDieren 2006) Any *tame* AEC with a monster model admits a generalisation of Morley’s categoricity theorem, with a caveat about the starting cardinality:

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\text{if it only has (up to isomorphism) } 1 \text{ member of cardinality } \lambda^+ \text{ for a sufficiently large } \lambda, \text{ then it only has 1 member (up to isomorphism) in each cardinality } \mu \geq \lambda^+.
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Here, an AEC is said to be *<\kappa*-tame if for any two distinct Galois 1-types over some \(M \in K\), there is a subset \(A\) of \(M\) of size less than \(\kappa\) over which they are already different.

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some $M \in \mathcal{K}$, there is a subset $A$ of $M$ of size less than $\kappa$ over which they are
already different. $\mathcal{K}$ is *tame* if it is *$< \kappa$-tame* for some $\kappa$. 
(Grossberg & VanDieren 2006) Any tame AEC with a monster model admits a generalisation of Morley’s categoricity theorem, with a caveat about the starting cardinality: if it only has (up to isomorphism) 1 member of cardinality $\lambda^+$ for a sufficiently large $\lambda$, then it only has 1 member (up to isomorphism) in each cardinality $\mu \geq \lambda^+$.

Here, an AEC is said to be $< \kappa$-tame if for any two distinct Galois 1-types over some $M \in \mathcal{K}$, there is a subset $A$ of $M$ of size less than $\kappa$ over which they are already different. $\mathcal{K}$ is tame if it is $< \kappa$-tame for some $\kappa$.

**Theorem (Boney 2014)**

*If there is a proper class of strongly compact cardinals, then every AEC is tame.*
Theorem (B.-T. & Rosický, using accessible categories perspective)

If for every cardinal $\mu$ there exists a $\mu$-strongly compact cardinal (also called $L_{\mu,\omega}$-compact), then every AEC is tame.
Improving Boney’s Theorem

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If for every cardinal $\mu$ there exists a $\mu$-strongly compact cardinal (also called $\mathcal{L}_{\mu,\omega}$-compact), then every AEC is tame.

Theorem (Boney & Unger)

The above is optimal: if every AEC is tame, then for every $\mu$ there exists a $\mu$-strongly compact cardinal.
The weaker large cardinals

Recall that a cardinal $\kappa$ is strongly compact if every $\kappa$-complete filter (on any set) extends to a $\kappa$-complete ultrafilter.

Clearly if $\kappa$ is strongly compact it is $\gamma$-strongly compact for every $\gamma \leq \kappa$. Also note that if $\kappa$ is $\gamma$-strongly compact and $\lambda > \kappa$ then $\lambda$ is also $\gamma$-strongly compact.

It is consistently a strictly weaker notion than strong compactness: Bagaria and Magidor have shown that the least $\aleph_1$-strongly compact cardinal may be singular.
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Embedding reformulation

Theorem (Bagaria & Magidor, 2014)

A cardinal $\kappa$ is $\gamma$-strongly compact if and only if for every $\alpha \geq \kappa$ there is an elementary embedding $j : V \to M$ such that

1. $\kappa \geq \text{crit}(j) \geq \gamma$,
2. there is a set $A \supseteq j''\alpha$ such that $A \in M$ and $M \models |A| < j(\kappa)$.
More accessible categories background: powerful subcategories and images

A powerful subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is a full subcategory that is closed under taking subobjects.
More accessible categories background: powerful subcategories and images

A *powerful* subcategory $\mathcal{A}$ of a category $C$ is a full subcategory that is closed under taking subobjects, i.e. objects that map monomorphically into objects of $\mathcal{A}$. 

Motivating example

The category $\text{FrAb}$ of free abelian groups is a powerful subcategory of the category $\text{Ab}$ of abelian groups. Note that $\text{FrAb}$ is also the image of the free abelian group functor $F: \text{Set} \to \text{Ab}$. For a functor $F: \mathcal{A} \to \mathcal{C}$, the powerful image of $F$ is the least powerful subcategory of $\mathcal{C}$ containing the image of $F$, that is, the full subcategory with objects given by the closure of $\text{Im}(F)$ under subobjects.
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For a functor $F : \mathcal{A} \to C$, the powerful image of $F$ is the least powerful subcategory of $C$ containing the image of $F$, that is, the full subcategory with objects given by the closure of $\text{Im}(F)$ under subobjects.
Accessible functors

A functor $F : K \to L$ between $\lambda$-accessible categories $K$ and $L$ is $\lambda$-accessible if it preserves $\lambda$-directed colimits (as discussed yesterday, under Vopěnka's Principle, this will automatically hold for sufficiently large $\lambda$).

Example

The free abelian group functor $F : Set \to Ab$ is $\aleph_0$-accessible.

Theorem (Makkai & Pare, 1989)

Suppose there is a proper class of strongly compact cardinals. Then the powerful image of any accessible functor is accessible.
Accessible functors

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Suppose for every $\gamma$ there is a $\gamma$-strongly compact cardinal. Then the powerful image of any accessible functor is accessible.
Improving on Makkai & Paré

Theorem (B.-T. & Rosický)

Suppose for every $\gamma$ there is a $\gamma$-strongly compact cardinal. Then the powerful image of any accessible functor is accessible.

Corollary (By work of Lieberman & Rosický)

Suppose for every $\gamma$ there is a $\gamma$-strongly compact cardinal. Then every AEC is tame.
Sketch of the proof of the Theorem

From the general theory of accessible categories, we can reduce the problem to showing that the powerful image is closed under $\kappa$-directed colimits, for $\kappa$ a $\gamma$-strongly compact cardinal.
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For the rest, see the blackboard.