Large cardinal axioms in category theory III

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Breaking news about Vopěnka's Principle!

For the purposes of this slide, let Ord denote partial order category for the (class of all) ordinals with the usual order. Another formulation of Vopěnka's Principle is

Vopěnka's Principle, Ord embedding formulation

There is no full embedding of Ord into the category of graphs.

The dual statement is called *Weak Vopěnka's Principle*

Weak Vopěnka's Principle

There is no full embedding of Ord^{op} into the category of graphs.

This is useful for (indeed equivalent to) various statements about accessible categories.

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Last year, Trevor Wilson showed:

Theorem

"Ord is Woodin" implies Weak Vopěnka's Principle. In particular, it is strictly weaker than Vopěnka's Principle.

He also has a claimed proof that Weak Vopěnka's Principle implies "Ord is Woodin", pinpointing the exact strength of Weak Vopěnka's Principle. But it's a much more involved argument that's still being refereed and which I haven't read yet; it's at arXiv:1907.00284.

Part III: Strong compactness and an application

Abstract Elementary Classes (AECs)

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Notation

In this talk, for a language $\mathcal{L}$ and $\mathcal{L}$-structures M and N, we'll write $M \subseteq_{\mathcal{L}} N$ to mean M is an $\mathcal{L}$-substructure of N, and $|M|$ for the underlying set of the structure M.

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3. For M and N in $\mathbf{K}$, if $M \leq N$ then $M \subseteq_{\mathcal{L}} N$.

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2. $\leq$ is a reflexive and transitive binary relation on $\mathbf{K}$.
3. For M and N in $\mathbf{K}$, if $M \leq N$ then $M \subseteq_{\mathcal{L}} N$.
4. Both $\mathbf{K}$ and $\leq$ are closed under isomorphism: if $M, N \in \mathbf{K}$, $M \leq N$ and $f : N \cong N'$, then $f``M$ and $N'$ are in $\mathbf{K}$, and $f``M \leq N$.

... continued

Abstract Elementary Classes (AECs)

5. (*Coherence axiom*) If $M_0, M_1, M_2 \in \mathbf{K}$, $M_0 \subseteq_{\mathcal{L}} M_1 \leq M_2$, and $M_0 \leq M_2$, then $M_0 \leq M_1$.
6. (*Tarki-Vaught chain axioms*) If δ is a limit ordinal and $\langle M_i \mid i < \delta \rangle$ is a $\leq$-increasing chain of members of $\mathbf{K}$, then
   1. $M_\delta := \bigcup_{i<\delta} M_i \in \mathbf{K}$,
   2. $M_i \leq M_\delta$ for all $i < \delta$, and
   3. if $N \in \mathbf{K}$ and $M_i \leq N$ for all $i < \delta$ then $M_\delta \leq N$.

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    3. if $N \in \mathbf{K}$ and $M_i \leq N$ for all $i < \delta$ then $M_\delta \leq N$.
7. (*Löwenheim-Skolem-Tarski axiom*) There is a cardinal $\mu \geq \|\mathcal{L}(\mathbf{K})\| + \aleph_0$ such that for every $M \leq \mathbf{K}$ and every $A \subseteq |M|$, there exists $M_0 \leq M$ in $\mathbf{K}$ such that $A \subseteq |M|$ and $\|M_0\| \leq \mu + \|A\|$. We define the Löwenheim-Skolem-Tarski number of $\mathbf{K}$, $LS(\mathbf{K})$, to be the least such μ.

Examples

- The class of all models of a sentence of first order logic, or of any countable fragment of $\mathcal{L}_{\omega_1,\omega}$, with the associated notion of elementary submodel as $\leq$.

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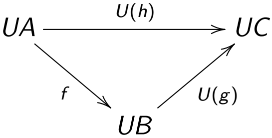
- Any class of models closed under elementary equivalence, with elementary submodel as $\leq$. Eg: Artinian rings.
- $\omega_1 + 1$ as **K**, usual ordinal $\leq$ as $\leq$.

Category theory perspective

Theorem (Lieberman and Rosický, 2016)

An AEC is precisely an accessible category $\mathcal{K}$ in which every morphism is a monomorphism, colimits exist for all directed diagrams, and there is a faithful functor U to **Set** *satisfying:*

- *U preserves directed colimits*
- *coherence: for any commutative diagram*



  there is an $\bar{f}$ in $\mathcal{K}$ such that $U(\bar{f}) = f$.
- *there is a canonical language $\mathcal{L}_U$ that can be associated with a functor U as above; if $f\colon UA \to UB$ is an $\mathcal{L}_U$-structure isomorphism, there is a $\bar{f}$ in $\mathcal{K}$ such that $f = U(\bar{f})$ (iso-fullness).*

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Here, an AEC is said to be $<\kappa$-*tame* if for any two distinct Galois 1-types over some $M \in \mathcal{K}$, there is a subset A of M of size less than κ over which they are already different. $\mathcal{K}$ is *tame* if it is $<\kappa$-tame for some κ.

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Theorem (Boney 2014)

If there is a proper class of strongly compact cardinals, then every AEC is tame.

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Theorem (B.-T. & Rosický, using accessible categories perspective)

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Theorem (Boney & Unger)

The above is optimal: if every AEC is tame, then for every μ there exists a μ-strongly compact cardinal.

The weaker large cardinals

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It is consistently a strictly weaker notion than strong compactness: Bagaria and Magidor have shown that the least $\aleph_1$-strongly compact cardinal may be singular.

Embedding reformulation

Theorem (Bagaria & Magidor, 2014)

A cardinal κ is γ-strongly compact if and only if for every $\alpha \geq \kappa$ there is an elementary embedding $j : V \to M$ such that

- $\kappa \geq \mathrm{crit}(j) \geq \gamma$,
- *there is a set $A \supseteq j``\alpha$ such that $A \in M$ and $M \vDash |A| < j(\kappa)$.*

More accessible categories background: powerful subcategories and images

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Motivating example

The category **FrAb** of free abelian groups is a powerful subcategory of the category **Ab** of abelian groups.

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The category **FrAb** of free abelian groups is a powerful subcategory of the category **Ab** of abelian groups.

Note that **FrAb** is also the image of the free abelian group functor F : **Set** $\rightarrow$ **Ab**.

For a functor $F : \mathcal{A} \rightarrow \mathcal{C}$, the *powerful image* of F is the least powerful subcategory of $\mathcal{C}$ containing the image of F, that is, the full subcategory with objects given by the closure of $\mathrm{Im}(F)$ under subobjects.

Accessible functors

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A functor $F : \mathcal{K} \to \mathcal{L}$ between λ-accessible categories $\mathcal{K}$ and $\mathcal{L}$ is *λ-accessible* if it preserves λ-directed colimits (as discussed yesterday, under Vopěnka's Principle, this will automatically hold for sufficiently large λ).

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Example

The free abelian group functor $F : \mathbf{Set} \to \mathbf{Ab}$ is $\aleph_0$-accessible.

Theorem (Makkai & Paré, 1989)

Suppose there is a proper class of strongly compact cardinals. Then the powerful image of any accessible functor is accessible.

Improving on Makkai & Paré

Theorem (B.-T. & Rosický)

Suppose for every γ there is a γ-strongly compact cardinal. Then the powerful image of any accessible functor is accessible.

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Theorem (B.-T. & Rosický)

Suppose for every γ there is a γ-strongly compact cardinal. Then the powerful image of any accessible functor is accessible.

Corollary (By work of Lieberman & Rosický)

Suppose for every γ there is a γ-strongly compact cardinal. Then every AEC is tame.

Sketch of the proof of the Theorem

From the general theory of accessible categories, we can reduce the problem to showing that the powerful image is closed under κ-directed colimits, for κ a γ-strongly compact cardinal.

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For the rest, see the blackboard.