ELEMENTARY EMBEDDINGS AND CORRECTNESS

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Kunen's inconsistency theorem can be generalized as follows:

Theorem 1. *Suppose $j : M \to N$ is a nontrivial elementary embedding between models of* ZFC *with the same ordinals. Then at least one of the following holds:*

(1) *There is $\alpha \in M$ such that $\langle \alpha, j(\alpha), j^2(\alpha), j^3(\alpha), \ldots \rangle \notin M$.*
(2) *There is $\alpha \in M$ such that $j[\alpha] \notin N$.*
(3) *Some ordinal is regular in M and singular in N.*

We investigate embeddings that come as close as possible to the boundaries imposed by the above theorem. One example shows that an inner model which is arbitrarily close to V may be self-embeddable.

Theorem 2. *Suppose $\kappa \leq \lambda$ are regular. κ is λ-supercompact if and only if there is a λ-closed transitive class M and a nontrivial elementary $j : M \to M$ with critical point κ and $\lambda < j(\kappa)$.*

We say $j : M \to N$ is *amenable* when alternative (2) above fails, which implies $M \subseteq N$. We give examples of such embeddings with various fixed-point properties, and also investigate the structure of the concrete categories of systems of models with the same ordinals and amenable maps between them. Let $\mathbf{AmOut}(M)$ be the category of models N with the same ordinals as M such that there is an amenable $j : M \to N$, with the arrows being all amenable embeddings between these objects.

Partial and linear orders are naturally represented as categories. A *pseudotree* is a partial order that is *linearly* ordered below any given element. We define a canonical countable pseudotree that contains every other countable pseudotree as a substructure. We show:

Theorem 3. *Suppose there is a countable transitive model of* ZFC *plus a measurable cardinal. Then there are many countable transitive $M \models$* ZFC *such that:*

(1) *For every linear order L, $\mathbf{AmOut}(M)$ has a subcategory isomorphic to L iff L is countable.*
(2) *For every countable partial order P, there is a subcategory of $\mathbf{AmOut}(M)$ isomorphic to P.*
(3) *There is an **incompatibility-preserving** injective functor from the "universal countable pseudotree" into $\mathbf{AmOut}(M)$.*

There are many natural questions about these categories which we do not know how to answer at present.

This is joint work with Sy Friedman.