Variations of reversibility

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Introduction

Introduction

- Generally speaking, we say that a structure $\mathbb{X}$ is reversible iff all its bijective endomorphisms are automorphisms
- The class of reversible structures contains, for example, compact Hausdorff and Euclidian topological spaces, linear oders, Boolean lattices, well founded posets with finite levels, tournaments, n-regular graphs, Henson graphs etc.
- extreme elements of $L_{\infty\omega}$-definable classes of interpretations under certain syntactical restrictions are reversible (Kurilić, M.)
- monomorphic (chainable) structures are reversible (Kurilić)
- Rado graph, the random poset, the ideal $\langle \mathrm{Fin}, \subseteq \rangle$, the lattices $\langle \mathbb{N}, | \rangle$ and $\langle \omega, | \rangle$ are non-reversible structures (Kurilić)
- Reversible structures have the property Cantor-Schröder-Bernstein (shorter CSB) for condensations (bijective homomorphisms)
- each class of reversible posets yields the corresponding class of reversible topological spaces if we observe topology generated by the basis consisting of principal ideals

Variations of reversibility

Variations of reversibility

Definition

We say that an L-interpretation $\rho \in \mathrm{Int}_L(X)$ is:

- *strongly reversible* iff $[\rho]_{\cong} = \{\rho\}$ (or, equivalently, $[\rho]_{\sim_c} = \{\rho\}$)
- *reversible* iff $[\rho]_{\cong}$ (or, equivalently, $[\rho]_{\sim_c}$) is an antichain in the Boolean lattice $\langle \mathrm{Int}_L(X), \subseteq \rangle$
- *weakly reversible* iff $[\rho]_{\cong}$ is a convex set in the Boolean lattice $\langle \mathrm{Int}_L(X), \subseteq \rangle$

Proposition

Let X be a nonempty set and L a relational language. Then we have:
(a) $\mathrm{sRev}_L(X) \subseteq \mathrm{Rev}_L(X) \subseteq \mathrm{wRev}_L(X)$;
(b) Strong reversibility, reversibility and weak reversibility are $\sim_c$-invariants (and, hence, $\cong$-invariants) on the set $\mathrm{Int}_L(X)$.

$\mathrm{sRev}_{L_b}(\omega) \subsetneq \mathrm{Rev}_{L_b}(\omega) \subsetneq \mathrm{wRev}_{L_b}(\omega) \subsetneq \mathrm{Int}_{L_b}(\omega)$

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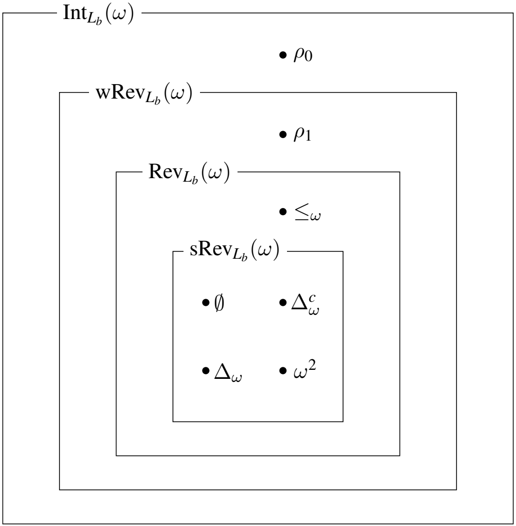

Strong reversibility

Strong reversibility

Strongly reversible relations are also known in the literature under the name of *constant relations*.

Theorem

Let X be a nonempty set and $L = \langle R_i : i \in I \rangle$ a relational language. For an interpretation $\rho \in \mathrm{Int}_L(X)$ the following conditions are equivalent:
(a) ρ is strongly reversible;
(b) ρ^c is strongly reversible;
(c) $\mathrm{Aut}(\rho) = \mathrm{Sym}(X)$;
(d) $\mathrm{Cond}(\rho) = \mathrm{Sym}(X)$;
(e) Each relations ρ_i, $i \in I$, is strongly reversible;
(f) Each relation ρ_i, $i \in I$, is a subset of the set X^{n_i} definable by an $L_\emptyset$-formula, without quantifiers and parameters.

As a consequence, we have that $\mathrm{sRev}_L(X)$ is a complete regular subalgebra of the complete Boolean algebra $\mathrm{Int}_L(X)$, and, in particuar, we have that

$$\mathrm{sRev}_{L_b}(X) = \{\emptyset, \Delta_X, \Delta_X^c, X^2\}.$$

Reversibility

Reversibility

Theorem

Let X be a nonempty set and $L = \langle R_i : i \in I \rangle$ a relational language. For an interpretation $\rho \in \mathrm{Int}_L(X)$ the following conditions are equivalent:
(a) ρ is reversible;
(b) ρ^c is reversible;
(c) $\mathrm{Aut}(\rho) = \mathrm{Cond}(\rho)$;
(d) $\mathrm{Cond}(\rho)$ is a subgroup of the symmetrical group $\mathrm{Sym}(X)$.

We have that $\mathrm{Fcf}_L(X) \subseteq \mathrm{Rev}_L(X)$, where

$$\mathrm{Fcf}_L(X) := \{\rho \in \mathrm{Int}_L(X) : \forall i \in I \ \ (|\rho_i| < \omega \vee |X^{n_i} \setminus \rho_i| < \omega)\}.$$

Therefore, if the set X is finite, we have that $\mathrm{Rev}_L(X) = \mathrm{Int}_L(X)$. Also, $\mathrm{fRev}_L(X) \subseteq \mathrm{Rev}_L(X)$, where

$$\mathrm{fRev}_L(X) := \{\rho \in \mathrm{Int}_L(X) : |\,\mathrm{Cond}(\rho)| < \omega\}.$$

Weak reversibility

Weak reversibility

We say that an interpretation $\rho \in \mathrm{Int}_L(X)$ has the *property Cantor-Schröder-Bernstein for condensations* iff whenever $f : \langle X, \rho\rangle \to \langle X, \sigma\rangle$ and $g : \langle X, \sigma\rangle \to \langle X, \rho\rangle$ are condensations, we have that $\rho \cong \sigma$, for arbitrary $\sigma \in \mathrm{Int}_L(X)$.

Theorem

Let X be a nonempty set and $L = \langle R_i : i \in I\rangle$ a relational language. For an interpretation $\rho \in \mathrm{Int}_L(X)$ the following conditions are equivalent:
(a) ρ is weakly reversible;
(b) ρ^c is weakly reversible;
(c) $[\rho]_{\cong} = [\rho]_{\sim_c}$;
(d) ρ has the property Cantor-Schröder-Bernstein for condensations.

Given $\rho \in \mathrm{Int}_L(X)$ let us define the following L-interpretation:

$$\rho^* := \bigcup \Big\{\sigma \in \mathrm{At}(\mathrm{Int}_L(X)^+) \cap \rho\!\downarrow : \rho \cong \rho \setminus \sigma\Big\}.$$

Reversibility vs. weak reversibility

Reversibility vs. weak reversibility

In particular, if $L = L_b$, then $\rho^* = \Big\{\langle x,y\rangle \in \rho : \rho \cong \rho \setminus \{\langle x,y\rangle\}\Big\}$.

Theorem

Let X be a nonempty set and $L = \langle R_i : i \in I\rangle$ a relational language. For an interpretation $\rho \in \mathrm{wRev}_L(X)$ we have:
(a) $\rho^* = \langle \emptyset : i \in I\rangle \iff \rho \in \mathrm{Rev}_L(X)$;
(b) $\forall i \in I \ \Big((\rho^*)_i \neq \emptyset \implies |(\rho^*)_i| > \omega\Big)$.

Consequently, we have that in the following classes of binary structures weak reversibility and reversibility are equivalent properties:

- equivalence relations and graphs
- dense partial orders and disjoint unions of chains
- trees having $< \omega$ maximal elements
- separative posets having $< \omega$ minimal elements
- lattices where each element (except, maybe, the largest) is $\wedge$-reducible
- lattices where each element (except, maybe, the smallest) is $\vee$-reducible

Characterization of some CSB structures

Characterization of some CSB structures

Theorem

Let $\sim$ be an equivalence relation on a set X and let $X/\sim = \{X_i : i \in I\}$ be the corresponding partition. Then the structure $\mathbb{X} := \langle X, \sim\rangle$ has the property CSB for condensations iff the sequence of cardinals $\langle |X_i| : i \in I\rangle$ is finite-to-one, or it is a reversible sequence of natural numbers.

For a sequence of ordinals $\langle \alpha_i : i \in I\rangle$, where $\alpha_i = \gamma_i + n_i$, let us define sets

$$I_\alpha := \{i \in I : \alpha_i = \alpha\}, \text{ for } \alpha \in \text{Ord}, \quad J_\gamma := \{j \in I : \gamma_j = \gamma\}, \text{ for } \gamma \in \text{Lim}_0 .$$

Theorem

Poset $\bigcup_{i\in I} \alpha_i$ has the property CSB for condensations iff exactly one of the following two cases holds:

(I) The sequence $\langle \alpha_i : i \in I\rangle$ is finite-to-one,

(II) There exists $\gamma := \max\{\gamma_i : i \in I\}$, for $\alpha \leq \gamma$ we have that $|I_\alpha| < \omega$, and the sequence of natural numbers $\langle n_i : i \in J_\gamma \setminus I_\gamma\rangle$ is reversible, but not finite-to-one.

Examples

Examples

Example

- Rado graph, the random poset, ideal $\langle \mathrm{Fin}, \subseteq \rangle$, lattices $\langle \mathbb{N}, | \rangle$ and $\langle \omega, | \rangle$ do not have the property CSB for condensations.
- The class $\mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega)$ contains various structures:
  1. If $\mathbb{X}_1 = \langle \omega, \rho_1 \rangle := \bigcup_\omega \mathbb{L}_2 \cup \bigcup_\omega \mathbf{1}$, then $\rho_1 \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega)$, and the structure $\mathbb{X}_1$ is a non-rooted tree.
  2. If $\mathbb{X}_2 = \langle \omega, \rho_2 \rangle := \mathbf{1} + \mathbb{X}_1$, then $\rho_2 \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega)$, and the structure $\mathbb{X}_2$ is a rooted tree.
  3. If $\mathbb{X}_3 = \langle \omega, \rho_3 \rangle := (\mathbb{A}_\omega + \mathbf{1}) \cup \bigcup_\omega \mathbf{1}$, then $\rho_3 \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega)$, and the structure $\mathbb{X}_3$ is a separative poset.
  4. If $\mathbb{X}_4 = \langle \omega, \rho_4 \rangle := \mathbf{1} + \left(\bigcup_\omega \mathbb{L}_4 \cup \bigcup_\omega \mathbb{B}_2 \right) + \mathbf{1}$, then $\rho_4 \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega)$, and the structure $\mathbb{X}_4$ is a lattice.
  5. The structure $\mathbb{X}_1$ is disconnected and $\mathbb{X}_1^c$ is connected.
  6. The structure $\mathbb{X}_2$ is bi-connected.

Properties of weakly reversible interpretations

Properties of weakly reversible interpretations

Proposition

Let X be a nonempty set and L a relational language. If $\rho \in \mathrm{wRev}_L(X) \setminus \mathrm{Rev}_L(X)$ we have:
(a) The interpretation ρ^* is not reversible;
(b) The interpretation $\rho \setminus \rho^*$ is not finitary reversible;
(c) $\rho \ncong \sigma$, and thus also $\rho \nsim_c \sigma$, for each $\sigma \subseteq \rho \setminus \rho^*$;
(d) If $\rho \setminus \rho^* \in \mathrm{Rev}_L(X)$, and if $L = L_n = \langle R \rangle$, where $\mathrm{ar}(R) = n$, then $\rho \cong \rho \setminus \sigma$, for each $\sigma \in [\rho^*]^{<\omega}$;
(e) If $\rho \setminus \rho^* \in \mathrm{sRev}_L(X)$, then $\rho^* \in \mathrm{wRev}_L(X)$;
(f) If $\rho^* \in \mathrm{wRev}_L(X)$ or $\rho \setminus \rho^* \in \mathrm{Rev}_L(X)$, then $(\rho^*)^* = \rho^*$.

Examples

Examples

Example

$\rho_k \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega)$, for $k \in \{1,2,3,4\}$.

1. If $\mathbb{X}_1 = \langle \omega, \rho_1 \rangle := \bigcup_\omega \mathbb{D}_2 \cup \bigcup_\omega \mathbf{1}$, then

$$\rho_1^* = \rho_1 \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega), \qquad \rho_1 \setminus \rho_1^* = \emptyset \in \mathrm{sRev}_{L_b}(\omega)$$

$$\mathrm{Cond}(\rho_1) = \mathrm{Cond}(\rho_1^*), \qquad (\rho_1^*)^* = \rho_1^*.$$

2. If $\mathbb{X}_2 = \langle \omega, \rho \rangle := \mathbb{G}_2 \cup \bigcup_\omega \mathbb{D}_2 \cup \bigcup_\omega \mathbf{1}$, then

$$\rho_2^* \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega), \qquad \rho_2 \setminus \rho_2^* \in \mathrm{Rev}_{L_b}(\omega) \setminus \mathrm{sRev}_{L_b}(\omega),$$

$$\mathrm{Cond}(\rho_2) \subsetneq \mathrm{Cond}(\rho_2^*), \qquad (\rho_2^*)^* = \rho_2^*.$$

Examples and open questions

Examples and open questions

3. If $\mathbb{X}_3 = \langle \omega, \rho_3 \rangle := \bigcup_\omega \mathbb{G}_2 \cup \bigcup_\omega \mathbb{D}_2$, then

$$\rho_3^* \notin \mathrm{wRev}_{L_b}(\omega), \qquad \rho_3 \setminus \rho_3^* \cong \rho_1 \in \mathrm{wRev}_{L_b}(\omega) \setminus \mathrm{Rev}_{L_b}(\omega),$$

$$\mathrm{Cond}(\rho_3) \subsetneq \mathrm{Cond}(\rho_3^*), \qquad \emptyset = (\rho_3^*)^* \subsetneq \rho_3^*.$$

4. If $\mathbb{X}_4 = \langle \omega, \rho_4 \rangle = \bigcup_\omega \mathbb{C}_3 \cup \bigcup_\omega \mathbb{D}_3$, then

$$\rho_4^* \notin \mathrm{wRev}_{L_b}(\omega), \qquad \rho_4 \setminus \rho_4^* \notin \mathrm{wRev}_{L_b}(\omega),$$

$$\mathrm{Cond}(\rho_4) \subsetneq \mathrm{Cond}(\rho_4^*), \qquad \emptyset = (\rho_4^*)^* \subsetneq \rho_4^*.$$

Here we encounter the following open questions:

1. Is there a $\rho \in \mathrm{wRev}_L(X) \setminus \mathrm{Rev}_L(X)$ such that $\rho^* \in \mathrm{wRev}_L(X)$ and $\rho \setminus \rho^* \notin \mathrm{Rev}_L(X)$?
2. Is there a $\rho \in \mathrm{wRev}_L(X) \setminus \mathrm{Rev}_L(X)$ such that $\rho^* \notin \mathrm{wRev}_L(X)$ and $\rho \setminus \rho^* \in \mathrm{Rev}_L(X)$?

On interpretations of arbitrary languages

On interpretations of arbitrary languages

Proposition

Let X be a nonempty set and $L = \langle R_i : i \in I \rangle$ a relational language. Then for an interpretation $\rho = \langle \rho_i : i \in I \rangle \in \mathrm{Int}_L(X)$ we have:
(a) The interpretation ρ is strongly reversible iff each relation ρ_i, $i \in I$, is strongly reversible;
(b) If relations ρ_i, $i \in I$, are reversible then the interpretation ρ is reversible;
(c) If there exists $i_0 \in I$ such that the relation ρ_{i_0} is weakly reversible, and such that relations ρ_i, $i \in I \setminus \{i_0\}$, are strongly reversible, then the interpretation ρ is weakly reversible.

If we substitute strong reversibility with reversibility in (c), the statement fails to be true.

Open questions

Open questions

Here we encounter some basic open questions that are still open. Namely, let $L = \langle R_1, R_2 \rangle$, where $\mathrm{ar}(R_1) = \mathrm{ar}(R_2) = 2$:

1. Is there a $\rho = \langle \rho_1, \rho_2 \rangle \in \mathrm{wRev}_L(X) \setminus \mathrm{Rev}_L(X)$ such that

$$\{\rho_1, \rho_2\} \cap \left(\mathrm{wRev}_{L_b}(X) \setminus \mathrm{Rev}_{L_b}(X) \right) = \emptyset?$$

2. Is there $\rho = \langle \rho_1, \rho_2 \rangle \in \mathrm{wRev}_L(X) \setminus \mathrm{Rev}_L(X)$ such that

$$\{\rho_1, \rho_2\} \cap \mathrm{sRev}_{L_b}(X) = \emptyset?$$

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