The Open Dihypergraph Dichotomy for Definable Subsets of Generalized Baire Spaces

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(in joint work with Philipp Schlicht)

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Generalized Baire spaces

Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The κ-Baire space ${}^{\kappa}\kappa$ is the set of functions $f : \kappa \to \kappa$, with the bounded topology: basic open sets are of the form

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κ-Borel sets: close the family of open subsets under intersections and unions of size $\leq \kappa$ and complementation.

κ-analytic (or $\mathbf{\Sigma}^1_1(\kappa)$) sets: continuous images of κ-Borel sets; equivalently: continuous images of closed sets.

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  i.e., there is a continuous embedding $f : {}^{\kappa}2 \to X$ such that $(f(x), f(y)) \in G$ for all distinct $x, y \in {}^{\kappa}2$.

$\mathrm{OGD}_\omega(X)$ for definable subsets X of ${}^\omega\omega$

Theorem (Feng (1993); Todorčević)

$\mathrm{OGD}_\omega(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* $X \subseteq {}^\omega\omega$.

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These results give the exact consistency strength of these statements.

A higher dimensional version

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. Let $X \subseteq {}^{\kappa}\kappa$ and let $2 \leq D \leq \kappa$. A D-dimensional dihypergraph is a set $H \subseteq {}^{D}X$ of non-constant sequences.

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$\mathrm{OGD}^{2}_{\kappa}(X)$ is equivalent to the open graph dichotomy $\mathrm{OGD}_{\kappa}(X)$.

Applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$

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They also obtain several dichotomies for the second level of the Borel hierarchy as special cases of $\mathrm{OGD}_\omega^\omega(X)$. For example:

Theorem (R. Carroy, B.D. Miller, D.T. Soukup (2018))
Let $X \subseteq {}^\omega\omega$. If $\mathrm{OGD}_\omega^\omega(X)$ holds, then X satisfies the Hurewicz dichotomy (i.e., either X is contained in a K_σ subset of ${}^\omega\omega$ or there is a closed set $Y \subseteq X$ homeomorphic to ${}^\omega\omega$).

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  Is OGA_{κ} consistent? If so, how does it influence the structure of the κ-Baire space?

Thank you!