The Open Dihypergraph Dichotomy for Definable Subsets of Generalized Baire Spaces

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Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. The generalized Baire space ${}^{\kappa}\kappa$ is the space of functions $\kappa \to \kappa$, equipped with the bounded topology (i.e. the topology generated by the set of cones determined by functions with bounded domain). The investigation of the topology and descriptive set theory of these spaces, and specifically of the uncountable generalizations of regularity properties, is an active area of research today.

The open graph dichotomy [1] for a given subset X of the Baire space ${}^{\omega}\omega$ is a generalization of the perfect set property for X which can also be viewed as a definable version of the Open Coloring Axiom restricted to X. Feng [1] showed that it is consistent relative to the existence of an inaccessible cardinal that the open graph dichotomy holds for all subsets of the Baire space ${}^{\omega}\omega$ that are definable from a countable sequence of ordinals. In [2], higher dimensional versions of the open graph dichotomy are introduced and several well-known dichotomy theorems for the second level of the Borel hierarchy are obtained as special cases of the ω-dimensional version.

We study the uncountable analogues of the open graph dichotomy and its higher dimensional versions for the generalized Baire space ${}^{\kappa}\kappa$. Given $X \subseteq {}^{\kappa}\kappa$ and $2 \leq \delta \leq \kappa$, we let $\mathrm{OGD}^{\delta}_{\kappa}(X)$ denote the following statement.

> Suppose H is a δ-dimensional box-open dihypergraph on X (i.e., $H \subseteq {}^{\delta}X$ is a set of non-constant sequences which is open in the box-topology on ${}^{\delta}X$).
>
> Then either H has a coloring with κ many colors, or there exists a continuous homomorphism from a certain "large" δ-dimensional dihypergraph $\mathbb{H}$ on ${}^{\kappa}\delta$ to H.

In the case of $\delta = 2$, the existence of such a continuous homomorphism is equivalent to the existence of a κ-perfect subset $Y \subseteq X$ such that $H{\restriction}Y$ is a complete graph. Therefore $\mathrm{OGD}^{2}_{\kappa}(X)$ is equivalent to (the κ-analogue of) the open graph dichotomy.

We extend Feng's above mentioned theorem to the generalized Baire space ${}^{\kappa}\kappa$ and also obtain higher dimensional versions of this result. Namely, we show that for any infinite cardinal κ with $\kappa^{<\kappa} = \kappa$, the following statements are consistent relative to (and are therefore equiconsistent with) the existence of an inaccessible cardinal above κ.

1. $\mathrm{OGD}^{\delta}_{\kappa}(X)$ holds for all $2 \leq \delta < \kappa$ and all subsets $X \subseteq {}^{\kappa}\kappa$ which are definable from a κ-sequence of ordinals.

2. If $X \subseteq {}^{\kappa}\kappa$ is definable from a κ-sequence of ordinals, then $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ holds restricted to a certain class of κ-dimensional box-open dihypergraphs H on X. This class includes those box-open dihypergraphs H on X such that $H = H' \cap {}^{\kappa}X$ for some $\mathbf{\Sigma}^1_1(\kappa)$ subset H' of ${}^{\kappa}({}^{\kappa}\kappa)$.

This is joint work with Philipp Schlicht.

References

[1] Q. Feng, Homogeneity for open partitions of pairs of reals, *Trans. Amer. Math. Soc.*, 339(2):659–684, 1993.

[2] R. Carroy, B.D. Miller, and D.T. Soukup, The open dihypergraph dichotomy and the second level of the Borel hierarchy, submitted.