Generalized Egorov's statement for ideals

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Preliminaries and motivation: Classical Egorov's Theorem

Theorem (Egorov), [4]

Given a sequence of Lebesgue measurable functions $\langle f_n \rangle_{n\in\omega}$, $f_n\colon [0,1] \to [0,1]$ which is pointwise convergent on $[0,1]$ and $\varepsilon > 0$, one can find a measurable set $A \subseteq [0,1]$ with $m(A) \geq 1 - \varepsilon$ such that the sequence converges uniformly on A.

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If $\mathrm{non}\mathcal{N} < \mathfrak{b}$, the generalized Egorov's statement holds.

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If non$\mathcal{N} = \mathfrak{d} = \mathfrak{c}$, the generalized Egorov's statement fails.
Also if there exists a $\mathfrak{c}$-Lusin set and non$\mathcal{N} = \mathfrak{c}$, the generalized Egorov's statement fails.

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Recall that a set L is a κ-**Lusin set** if for any meagre set X, $|L \cap X| < \kappa$, but $|L| \geq \kappa$.

Convergence with respect to an ideal

Sequence convergence with respect to I

Given an ideal I on ω and a sequence $\langle x_n \rangle_{n\in\omega} \in \mathbb{R}^\omega$ we say that the sequence **converges** to a point $x \in \mathbb{R}$ with respect to I ($x_n \to_I x$) if for every $\varepsilon > 0$,

$$\{n \in \omega \colon |x_n - x| > \varepsilon\} \in I.$$

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I^*-convergence

A sequence $\langle x_n \rangle_{n\in\omega} \in \mathbb{R}^\omega$ I^*-converges to a point $x \in \mathbb{R}$ ($x_n \to_{I^*} x$) if there exists $C \in I$ such that the sequence $\langle x_n \rangle_{n\in(\omega\setminus C)}$ converges to x in the usual sense.

Convergence with respect to an ideal

Convergence of a sequence of functions with respect to I

We get different notions of convergence of a sequence $\langle f_n \rangle_{n\in\omega}$ of functions $[0,1] \to [0,1]$ on $A \subseteq [0,1]$ with respect to an ideal I on ω, which were introduced in [1] and [3]:

pointwise ideal, $f_n \to_I f$ if and only if

$$\forall_{\varepsilon>0}\forall_{x\in A}\,\{n\in\omega\colon |f_n(x)-f(x)|\geq\varepsilon\}\in I,$$

uniform ideal, $f_n \rightrightarrows_I f$ if and only if

$$\forall_{\varepsilon>0}\exists_{B\in I}\forall_{x\in A}\,\{n\in\omega\colon |f_n(x)-f(x)|\geq\varepsilon\}\subseteq B.$$

Convergence with respect to an ideal, continued

I^*-convergence of a sequence of functions

In this approach we get the following notions of convergence of a sequence $\langle f_n \rangle_{n\in\omega}$ of functions $[0,1] \to [0,1]$ on $A \subseteq [0,1]$:

I^*-pointwise, $f_n \to_{I^*} f$ if and only if for all $x \in A$, there exists $M = \{m_i \colon i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i}(x) \to f(x)$,

I^*-uniform, $f_n \rightrightarrows_{I^*} f$ if and only if there exists $M = \{m_i \colon i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i} \rightrightarrows f$ on A.

Convergence with respect to an ideal, continued

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J, I-convergence

The above notions can be further generalized. Let $J \subseteq I$ be ideals. If $A \subseteq [0,1]$ and $\langle f_n \rangle_{n \in \omega}$ is a sequence of functions $[0,1] \to [0,1]$, we have the following notions of convergence.

(J, I)-pointwise, $f_n \to_{J,I} f$ if and only if for all $x \in A$, there exists $N \in I$ such that for all $\varepsilon > 0$,

$$\{n \in \omega \colon |f_n(x) - f(x)| \geq \varepsilon\} \in J \vee \langle N \rangle,$$

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Notice that $\to_{I,I} = \to_I$ and $\rightrightarrows_{I,I} = \rightrightarrows_I$. Moreover, $\to_{\mathrm{Fin},I} = \to_{I^*}$, and $\rightrightarrows_{\mathrm{Fin},I} = \rightrightarrows_{I^*}$.

Convergence with respect to an ideal, continued

Therefore we have the following implications between notions of convergence for ideals $J \subseteq I$.

$$\begin{array}{ccccccc}
\to_{\mathrm{Fin}} & \Rightarrow & \to_{I^*} & \Rightarrow & \to_{J,I} & \Rightarrow & \to_I \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\rightrightarrows_{\mathrm{Fin}} & \Rightarrow & \rightrightarrows_{I^*} & \Rightarrow & \rightrightarrows_{J,I} & \Rightarrow & \rightrightarrows_I
\end{array}$$

Egorov's statement for ideals: countably generated ideals

An ideal I is countably generated (satisfies the chain condition) if there exists a sequence $\langle C_i \rangle_{i \in \omega}$ of elements of I such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$.

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Proposition

If I is a countably generated ideal on ω, and $f_n \colon [0,1] \to [0,1]$, $n \in \omega$ are Lebesgue-measurable functions such that $f_n \to_I 0$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteq [0,1]$ such that $m(B) \leq \varepsilon$ and $f_n \rightrightarrows_I 0$ on $[0,1] \setminus B$.

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But there are only two countably generated ideals on ω up to isomorphism...

Egorov's statement for ideals: Fin^α

Given an ideal I on ω and a sequence $\langle I_n \rangle_{n\in\omega}$ of ideals on ω, let $I\text{-}\prod_{n\in\omega} I_n$ be the following ideal. For any $A \subseteq \omega^2$,

$$A \in I\text{-}\prod_{n\in\omega} I_n \Leftrightarrow \{n \in \omega \colon A_{(n)} \notin I_n\} \in I,$$

where $A_{(n)} = \{m \in \omega \colon \langle n, m\rangle \in A\}$. If $I_n = J$ for any $n \in \omega$, we denote $I\text{-}\prod_{n\in\omega} I_n$ by $I \times J$.

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Fix a bijection $b\colon\omega^2\to\omega$ and a bijection $a_\beta\colon\omega\to\beta\setminus\{0\}$ for any limit $\beta<\omega_1$. The **ideals** $\mathbf{Fin}^{\alpha}$, $\alpha<\omega_1$, are defined inductively in the following way. Let $\mathrm{Fin}^1=\mathrm{Fin}$ be the ideal of finite subsets of ω. We set

$$\mathrm{Fin}^{\alpha+1}=\{b[A]\colon A\in\mathrm{Fin}\times\mathrm{Fin}^{\alpha}\},$$

and for limit $\beta<\omega_1$, let

$$\mathrm{Fin}^{\beta}=\left\{b[A]\colon A\in\mathrm{Fin}\text{-}\prod_{i\in\omega}\mathrm{Fin}^{a_\beta(i)}\right\}.$$

Egorov's statement for ideals: Fin^α

Theorem (N. Mrożek, 2010), [8]

If $I = \mathrm{Fin}^\alpha$ for $\alpha < \omega_1$, and $f_n \colon [0,1] \to [0,1]$, $n \in \omega$ are Lebesgue-measurable functions such that $f_n \to_I 0$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteq [0,1]$ such that $m(B) \leq \varepsilon$ and $f_n \rightrightarrows_I 0$ on $[0,1] \setminus B$.

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An ideal I is a **P-ideal** if for any sequence $\langle A_i \rangle_{i\in\omega} \in I^\omega$ of mutually disjoint sets, there exists a sequence $\langle B_i \rangle_{i\in\omega}$ such that $A_i \triangle B_i$ is finite for all $i \in \omega$, and $\bigcup_{i\in\omega} B_i \in I$.

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By the well-known result of Solecki ([11]) if I is an analytic P-ideal, then $I = \mathrm{Exh}(\phi)$, where ϕ is a lower semicontinuous submeasure.

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By the well-known result of Solecki ([11]) if I is an analytic P-ideal, then $I = \mathrm{Exh}(\phi)$, where ϕ is a lower semicontinuous submeasure.
A function $\phi \colon 2^\omega \to [0, \infty]$ is a lower semicontinuous submeasure if it satisfies the following conditions:

(1) $\phi(\varnothing) = 0$,

(2) $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$, for any $A, B \subseteq \omega$,

(3) $\phi(A) = \lim_{n\to\omega} \phi(A \cap n)$, for any $A \subseteq \omega$,

and,

$$\mathrm{Exh}(\phi) = \{A \subseteq \omega \colon \lim_{n\to\infty} \phi(A \setminus n) = 0\}.$$

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Egorov's statement for ideals: analytic P-ideals

Theorem (N. Mrożek, 2009), [7]

If I is an analytic P-ideal, and $f_n\colon [0,1] \to [0,1]$, $n \in \omega$ are Lebesgue-measurable functions such that $f_n \to_I 0$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteq [0,1]$ such that $m(B) \le \varepsilon$ and $f_n \twoheadrightarrow_I 0$ on $[0,1] \setminus B$.

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Theorem (N. Mrożek, 2009), [7]

If I is an analytic P-ideal which is not countably generated and non-pathological. Then there exists a sequence $f_n\colon [0,1] \to [0,1]$, $n \in \omega$ of Lebesgue-measurable functions such that $f_n \to_I 0$ and $\varepsilon > 0$, such that for every a measurable set $B \subseteq [0,1]$ with $m(B) \leq \varepsilon$ and $f_n \not\twoheadrightarrow_I 0$ on $[0,1] \setminus B$.

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A submeasure ϕ is non-pathological if it is equal to pointwise supremum of measures dominated by itself. An analityc P-ideal is non-pathological if $I = \mathrm{Exh}(\phi)$ for a non-pathological lower semicontinuous submeasure ϕ.

The method

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Crucial lemma (MK), [5]

Assume that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$. Let $\Phi \in (\omega^\omega)^{[0,1]}$. Then for any $\varepsilon > 0$, there exists $A \subseteq [0,1]$ such that $m^*(A) \geq 1 - \varepsilon$ and Φ is bounded on A.

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Assume that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$. Notice that this statement holds for example in a model obtained by $\aleph_2$-iteration with countable support of Laver forcing (see e.g. [2]). Also it can be easily proven that under this assumption there exists a set $Y \subseteq [0,1]$ of cardinality less that $\mathfrak{b}$ such that $m^*(Y) = 1$. Indeed, if $N \subseteq [0,1]$ is a set of positive outer measure with $|N| < \mathfrak{b}$, then let $Y = \{x + y \colon x \in N, y \in \mathbb{Q}\}$, where $+$ denotes addition modulo 1. Then Y has outer measure 1 under the Zero-One Law.

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Therefore, every function $\varphi \colon [0,1] \to \omega^\omega$ maps Y onto a K_σ-set, where K_σ denotes the σ-ideal of subsets of ω^ω generated by the compact (equivalently bounded) sets. We get that $\Phi[Y] \in K_\sigma$. Assume that $\Phi[Y] \subseteq \bigcup_{n\in\omega} B_n$ with each B_n bounded. Let $A_n = \Phi^{-1}[\bigcup_{i=0}^{n} B_i]$. Therefore, $\Phi[A_n]$ is bounded, and for any $\varepsilon > 0$, there exists $n \in \omega$ such that $m^*(A_n) \geq 1 - \varepsilon$. □

Witness function o

For a sequence of functions $f_n : [0,1] \to [0,1]$ and subsets $A \subseteq [0,1]$, we consider a notion of convergence $f_n \looparrowright f$ on A. We assume that if $B \subseteq A$ and $f_n \looparrowright f$ on A, then $f_n \looparrowright f$ on B.

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We say that a function $o\colon X \to P$ from a set X into a partially ordered set P is cofinal if for every $p \in P$ there exists $x \in X$ such that $p \leq o(x)$.

The positive theorem

Theorem (MK), [5]

Assume that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$, and $H^{\rightrightarrows}(\mathcal{F}, \not\to)$. Then for any $\langle f_n \rangle_{n\in\omega} \in \mathcal{F}$ and any $\varepsilon > 0$, there exists $A \subseteq [0,1]$ such that $m^*(A) \geq 1 - \varepsilon$ and $f_n \not\to 0$ on A.

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Proof: Apply the crucial lemma for $o(\langle f_n \rangle_{n\in\omega})$ given by $H^{\rightrightarrows}(\mathcal{F}, \not\to)$. □

The negative theorem

There exists a model of ZFC in which $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$, and there exists $\mathfrak{c}$-Lusin set. It suffices to iterate $\aleph_2$-times Cohen forcing with finite supports over a model of GCH (see [2, Model 7.5.8 and Lemma 8.2.6]).

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Proof: Again, we generalize some arguments of Pinciroli (see [9]). Let $Z \subseteq \omega^\omega$ be a $\mathfrak{c}$-Lusin set. Since every compact set is meagre in ω^ω, every K_σ set is also meagre. Therefore, if $A \subseteq Z$ is a K_σ set, then $|A| < \mathfrak{c}$. Let $o : \mathcal{F} \to (\omega^\omega)^{[0,1]}$ be a cofinal function given by $H^{\Leftarrow}(\mathcal{F}, \not\to)$. Let φ be a bijection between $[0,1]$ and Z. Finally, let $\langle f_n \rangle_{n\in\omega} = F \in \mathcal{F}$ be such that $o(F) \geq \varphi$.

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To get a contradiction, assume that for every $i \in \omega$, there exists $A_i \subseteq [0,1]$ such that $m^*(A_i) \geq 1 - 1/2^i$ and $f_n \not\to 0$ on A_i. Let $A = \bigcup_{i\in\omega} A_i$. For any $i \in \omega$, $o(F)[A_i]$ is bounded because $f_n \not\to 0$ on A_i, and so $\varphi[A_i]$ is bounded since $o(F) \geq \varphi$. Therefore, $\varphi[A] \in K_\sigma$ and $|A| = |\varphi[A]| < \mathfrak{c}$ because $\varphi[A] \subseteq Z$.

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Fin ideal

Let $\langle f_n \rangle_{n\in\omega}$ be such that $f_n \to 0$. Set $\varepsilon_n = 1/2^n$, $n \in \omega$. Consider $\mathcal{F} = \{\langle f_n \rangle_{n\in\omega} : \forall_{n\in\omega} f_n : [0,1] \to [0,1] \wedge f_n \to 0\}$ and $\looparrowright = \rightrightarrows$. Define $o : \mathcal{F} \to (\omega^\omega)^{[0,1]}$ in the following way. For $F = \langle f_n \rangle_{n\in\omega}$, let

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Theorem (R. Pinciroli, 2006), [9]

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Let from now on $\mathcal{F}_{\rightsquigarrow} = \{\langle f_n \rangle_{n\in\omega} : f_n \rightsquigarrow 0\}$ for a notion of convergence $\rightsquigarrow$.

And countably generated ideals

Assume that I is countably generated, and fix sets $\langle C_i \rangle_{i\in\omega}$ such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$. We can assume that $C_{i+1} \setminus C_i \neq \varnothing$ for all $i \in \omega$.

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If $F = \langle f_n \rangle_{n\in\omega}$, $f_n \to_I 0$, we define

$$(o_{\langle C_i \rangle} F)(x)(n) = \min\left\{k \in \omega\colon \left\{m \in \omega\colon f_m(x) > \frac{1}{2^n}\right\} \subseteq C_k\right\}.$$

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If $A \subseteq [0,1]$, then $f_n \rightrightarrows_I 0$ on A if and only if $(o_{\langle C_i \rangle} F)[A]$ is bounded, and so $H^{\rightrightarrows}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$ holds.

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If $A \subseteq [0,1]$, then $f_n \rightrightarrows_I 0$ on A if and only if $(o_{\langle C_i \rangle} F)[A]$ is bounded, and so $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$ holds.
Also, without a loss of generality we can assume that $\varphi(x)$ is increasing for all $x \in [0,1]$. Let $x \in [0,1]$. Let $f_j(x) = 1/2^n$ if and only if

$$j \in C_{\varphi(x)(n+1)} \setminus C_{\varphi(x)(n)}.$$

Thus, $H^{\Leftarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$ holds.

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Proposition

Assume that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$. Let I be any countably generated ideal, and let $\varepsilon > 0$. Let $F = \langle f_n \rangle_{n\in\omega}$, $f_n \colon [0,1] \to [0,1]$, for $n \in \omega$ be such that $f_n \to_I 0$. Then there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \rightrightarrows_I 0$ on A.

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Also for I^* convergence

Let $F = \langle f_n \rangle_{n\in\omega}$ be such that $f_n \to_{I^*} 0$. Let $F = \langle f_n \rangle_{n\in\omega}$ be such that $f_n \to_{I^*} 0$. For $x \in [0,1]$ define $o_{\langle C_i \rangle}(F)(x) = \psi \in \omega^\omega$ by

$$\psi(0) = \min \left\{ n \in \omega : \langle f_m \rangle_{m \in \omega \setminus C_n} \to 0 \right\},$$

$$\psi(n) = \min \left\{ m \in \omega : \forall_{\substack{l \in \omega \setminus C_{\psi(0)} \\ l > m}} f_l(x) < \frac{1}{2^n} \right\}, \quad n > 0.$$

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Analytic P-ideals

Fix ϕ such that $I = \text{Exh}(\phi)$. Notice that since I is a proper ideal, $\lim_{i\to\infty} \phi(\omega \setminus i) > 0$. If $\lim_{i\to\infty} \phi(\omega \setminus i) < \infty$, let

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Let I be an analytic P-ideal. Then, $f_n \twoheadrightarrow_I 0$ on $A \subseteq [0,1]$ if and only if $(o_\phi(\langle f_n \rangle_{n\in\omega}))[A]$ is bounded in ω^ω. In particular, $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds.

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Proof: By definition, $f_n\twoheadrightarrow_I 0$ on $A \Leftrightarrow$ for any $n\in\omega$, there exists $k\in\omega$ such that for all $x\in A$, $\phi(\{m\in\omega\colon f_m(x)\geq\varepsilon_n\}\setminus k) < \varepsilon_n \Leftrightarrow$ there exists a sequence $\langle k_n\rangle_{n\in\omega}$ of natural numbers such that for any $n\in\omega$ and $x\in A$, $\phi(\{m\in\omega\colon f_m(x)\geq\varepsilon_n\}\setminus k_n) < \varepsilon_n \Leftrightarrow$ for all $x\in A$, $(o_\phi F)(x)(n)\leq k_n$. □

Analytic P-ideals, continued

Theorem (MK),[5]

Assume that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$. Let I be any analytic P-ideal, $\varepsilon > 0$, and let $F = \langle f_n \rangle_{n\in\omega}$, $f_n \colon [0,1] \to [0,1]$ for $n \in \omega$, be such that $f_n \to_I 0$. Then there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \twoheadrightarrow_I 0$ on A

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$H^{\Leftarrow}(\mathcal{F}_{\rightarrow_I}, \twoheadrightarrow_I)$ holds.

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$H^{\Leftarrow}(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds.

Proof: Fix $x \in [0,1]$. Notice that $\phi(\omega \setminus n)$ is a decreasing sequence with limit greater or equal to $2\varepsilon_0 > 0$, so $\phi(\omega \setminus n) \geq 2\varepsilon_0 > 0$ for any $n \in \omega$. Therefore, for each $m, n \in \omega$, there exists $k > n$ such that $\phi(k \setminus n) > \varepsilon_m$. Let $\langle k_i \rangle_{i\in\omega}$ be an increasing sequence such that $k_0 = 0$ and $\phi(k_{i+1} \setminus \varphi(x)(i)) > \varepsilon_i$, $i \in \omega$. Set $f_j(x) = \varepsilon_i$ if $k_i \leq j < k_{i+1}$. Then $f_m(x) \geq \varepsilon_n$ if and only if $m < k_{n+1}$. Hence, if $\phi(\{m \in \omega \colon f_m(x) \geq \varepsilon_n\} \setminus k) < \varepsilon_n$, then $k \geq \varphi(x)(n)$, so $(o_\phi F)(x)(n) \geq \varphi(x)(n)$ for any $n \in \omega$. □

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Proof: Fix $x \in [0,1]$. Notice that $\phi(\omega \setminus n)$ is a decreasing sequence with limit greater or equal to $2\varepsilon_0 > 0$, so $\phi(\omega \setminus n) \geq 2\varepsilon_0 > 0$ for any $n \in \omega$. Therefore, for each $m, n \in \omega$, there exists $k > n$ such that $\phi(k \setminus n) > \varepsilon_m$. Let $\langle k_i \rangle_{i \in \omega}$ be an increasing sequence such that $k_0 = 0$ and $\phi(k_{i+1} \setminus \varphi(x)(i)) > \varepsilon_i$, $i \in \omega$. Set $f_j(x) = \varepsilon_i$ if $k_i \leq j < k_{i+1}$. Then $f_m(x) \geq \varepsilon_n$ if and only if $m < k_{n+1}$. Hence, if $\phi(\{m \in \omega \colon f_m(x) \geq \varepsilon_n\} \setminus k) < \varepsilon_n$, then $k \geq \varphi(x)(n)$, so $(o_\phi F)(x)(n) \geq \varphi(x)(n)$ for any $n \in \omega$. □

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Fin^{α} ideals

Let $\mathcal{F}_\alpha = \mathcal{F}_{\to_{\mathrm{Fin}^\alpha}}$.

Fin$^\alpha$ ideals

Let $\mathcal{F}_\alpha = \mathcal{F}_{\to_{\mathrm{Fin}^\alpha}}$.
Fix a bijection $b\colon \omega^2 \to \omega$ and a bijection $a_\beta\colon \omega \to \beta \setminus \{0\}$ for any limit $\beta < \omega_1$. We define $o_\alpha\colon \mathcal{F}_\alpha \to (\omega^\omega)^{[0,1]}$ in the following way. Let $\varepsilon_n = 1/2^n$ for $n \in \omega$, and let

$$\mathcal{F}^n_\alpha = \{\langle f_n\rangle_{n\in\omega} : \forall_{n\in\omega} f_n : [0,1] \to [0,1] \land \forall_{x\in[0,1]}\{q \in \omega\colon f_q(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha\}.$$

First, define $o^n_\alpha\colon \mathcal{F}^n_\alpha \to (\omega^\omega)^{[0,1]}$, $n \in \omega, 0 < \alpha < \omega_1$, by induction on α. Let

$$M_{1,n,x} = \min\{p \in \omega\colon \forall_{q\geq p} f_q(x) < \varepsilon_n\},$$

and let

$$(o^n_1 F)(x)(k) = M_{1,n,x}$$

be a constant sequence. Given o^n_α, let

$$M_{\alpha+1,n,x} = \min\left\{p \in \omega\colon \forall_{q\geq p}\{m \in \omega\colon f_{b(q,m)}(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha\right\},$$

and

$$(o^n_{\alpha+1}F)(x)(k) = \begin{cases} M_{\alpha+1,n,x} & \text{for } k = b(p,q),\\ & p < M_{\alpha+1,n,x}+1, q \in \omega,\\ \left(o^n_\alpha \left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega}\right)(x)(q), & \text{for } k = b(p,q),\\ & p \geq M_{\alpha+1,n,x}+1, q \in \omega. \end{cases}$$

This definition is correct, since $\left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega} \in \mathcal{F}^n_\alpha$ for $p \geq M_{\alpha+1,n,x}+1$.

Fin^{α} ideals, continued

Moreover, for limit $\beta < \omega_1$, let

$$M_{\beta,n,x} = \min\left\{p \in \omega\colon \forall_{q \geq p}\{m \in \omega\colon f_{b(q,m)}(x) \geq \varepsilon_n\} \in \mathrm{Fin}_{a_\beta(q)}\right\}$$

and

$$(o^n_\beta F)(x)(k) = \begin{cases} M_{\beta,n,x} & \text{for } k = b(p,q), \\ & p < M_{\beta,n,x}+1, q \in \omega, \\ \left(o^n_{a_\beta(p-1)} \left\langle f_{b(p-1,r)} \right\rangle_{r\in\omega}\right)(x)(q), & \text{for } k = b(p,q), \\ & p \geq M_{\beta,n,x}+1, q \in \omega. \end{cases}$$

This definition is correct, since, for each $p \geq M_{\beta,n,x}+1$, $\left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega} \in \mathcal{F}^n_{a_\beta(p-1)}$.

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Moreover, for limit $\beta < \omega_1$, let

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This definition is correct, since, for each $p \geq M_{\beta,n,x}+1$, $\left\langle f_{b(p-1,r)} \right\rangle_{r\in\omega} \in \mathcal{F}^n_{a_\beta(p-1)}$.
Notice that $\mathcal{F}_\alpha \subseteq \mathcal{F}^n_\alpha$, for any $n \in \omega$. Therefore, finally let

$$(o_\alpha F)(x)(k) = (o^n_\alpha F)(x)(m),$$

for $k = b(n,m)$, $n, m \in \omega$.

Fin^α ideals, continued

The construction is done in such a way that $H^{\Rightarrow}(\mathcal{F}_\alpha, \rightrightarrows_{\mathsf{Fin}^\alpha})$ holds.

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Assume that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$. Let $0 < \alpha < \omega_1$, and let $\varepsilon > 0$ and $F = \langle f_n \rangle_{n\in\omega}$, $f_n \colon [0,1] \to [0,1]$ for $n \in \omega$, with $f_n \to_{\mathrm{Fin}^\alpha} 0$. Then there exists $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A.

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Also,

Theorem (MK),[5]

Assume that $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Let $0 < \alpha < \omega_1$. Then there exist $\langle f_n \rangle_{n\in\omega} \in \mathcal{F}_\alpha$ and $\varepsilon > 0$ such that for all $A \subseteq [0,1]$ with $m^*(A) \geq 1 - \varepsilon$, $f_n \not\rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A.

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Sketch of the proof: It is enough to take

$$(o_\alpha F)(x)(n) = M_{\alpha,n,x}.$$

Other properties

A mapping $o\colon \mathcal{F} \to (\omega^\omega)^{[0,1]}$ is said to be measurability preserving, if for any sequence of measurable functions $\langle f_n \rangle_{n\in\omega} \in \mathcal{F}$, $o(f)$ is measurable as well.

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($\bar{H}^{\Rightarrow}(\mathcal{F}, \nrightarrow)$) There exists $o : \mathcal{F} \to (\omega^\omega)^{[0,1]}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq [0,1]$ if $o(F)[A]$ is bounded in $(\omega^\omega, \leq^*)$, then $F \nrightarrow 0$ on A.

($\bar{H}^{\Leftarrow}(\mathcal{F}, \nrightarrow)$) There exists $o : \mathcal{F} \to (\omega^\omega)^{[0,1]}$ which is cofinal (with respect to $\leq$) such that for every $F \in \mathcal{F}$ and every $A \subseteq [0,1]$, if $F \nrightarrow 0$ on A, then $o(F)[A]$ is bounded in $(\omega^\omega, \leq^*)$.

($M^{\Rightarrow}(\mathcal{F}, \nrightarrow)$) There exists measurability preserving $o : \mathcal{F} \to (\omega^\omega)^{[0,1]}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq [0,1]$ if $o(F)[A]$ is bounded in $(\omega^\omega, \leq)$, then $F \nrightarrow 0$ on A.

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Other properties, continued

We get that

$$
\begin{array}{ccc}
\bar{H}^{\Rightarrow}(\mathcal{F},\looparrowright) & \Rightarrow & H^{\Rightarrow}(\mathcal{F},\looparrowright) \\
\Uparrow & & \Uparrow \\
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It is also easy to get the following Corollary

Corollary (M. Repický), [10]

Assume that $M^{\Rightarrow}(\mathcal{F}_{\rightsquigarrow}, \looparrowright)$ holds. Then for every sequence of measurable functions $F = \langle f_n \rangle_n \in \omega \in \mathcal{F}_{\rightsquigarrow}$, and $\varepsilon > 0$, there exists a measurable set $A \subseteq [0, 1]$ such that $m(A) \geq 1 - \varepsilon$, and $f \looparrowright 0$ on A.

Other properties, continued

We get that

$$\begin{array}{ccccccc} \bar{H}^{\Rightarrow}(\mathcal{F}, \not\rightarrow) & \Rightarrow & H^{\Rightarrow}(\mathcal{F}, \not\rightarrow) & \quad & H^{\Leftarrow}(\mathcal{F}, \not\rightarrow) & \Rightarrow & \bar{H}^{\Leftarrow}(\mathcal{F}, \not\rightarrow) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ \bar{M}^{\Rightarrow}(\mathcal{F}, \not\rightarrow) & \Rightarrow & M^{\Rightarrow}(\mathcal{F}, \not\rightarrow) & & M^{\Leftarrow}(\mathcal{F}, \not\rightarrow) & \Rightarrow & \bar{M}^{\Leftarrow}(\mathcal{F}, \not\rightarrow) \end{array}$$

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Also the negative theorem can be slightly refined

Corollary (M. Repický), [10]

Assume that $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. If $\bar{H}^{\Leftarrow}(\mathcal{F}_{\rightsquigarrow}, \not\rightarrow)$ holds, then there exist $\langle f_n \rangle_{n\in\omega} \in \mathcal{F}_{\rightsquigarrow}$ and $\varepsilon > 0$ such that for all $A \subseteq [0, 1]$ with $m^*(A) \geq 1 - \varepsilon$, $f_n \not\not\rightarrow 0$ on A

Repický's results

In [10] the property $\bar{H}^{\Leftarrow}(\mathcal{F}_{\rightsquigarrow}, \looparrowright)$, where $\rightsquigarrow$ and $\looparrowright$ are various notions of convergence with respect to I is considered. In particular, it is proven that if $\rightsquigarrow$ is any notion of convergence weaker than $\to$, and $\looparrowright$ is stronger than $\rightrightarrows_I \cup \xrightarrow{QN}_{I^*}$, then $\bar{H}^{\Leftarrow}(\mathcal{F}_{\rightsquigarrow}, \looparrowright)$ holds.

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Actually, the function obtained in the proof of this observation witnesses $\bar{M}^{\Leftarrow}(\mathcal{F}_{\rightsquigarrow}, \looparrowright)$, and we have the following.

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Actually, the function obtained in the proof of this observation witnesses $\bar{M}^{\Leftarrow}(\mathcal{F}_{\rightsquigarrow}, \looparrowright)$, and we have the following.

Corollary (MK),[6]

Assume that I is an ideal on ω, and $\rightsquigarrow$ is any notion of convergence weaker than $\to$, and $\looparrowright$ is stronger than $\rightrightarrows_I \cup \xrightarrow{QN}_{I^*}$, then $\bar{M}^{\Leftarrow}(\mathcal{F}_{\rightsquigarrow}, \looparrowright)$ holds.

Repický's results, continued

$\mathcal{I}$:	$M^{\rightrightarrows}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$	$H^{\rightrightarrows}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$	$M^{\rightrightarrows}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$	$H^{\rightrightarrows}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$
Fin $\in$	✓	✓	✓	✓
$B \subseteq \omega$ is coinfinite, then $\langle B \rangle \in$	✓	✓	✓	✓
downward $\leq_{RK}$ closed	✓	✓		
downward $\leq_{RB}$ closed	✓	✓	✓	✓
$\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, then $b\left[\sum_{n\in\omega} I_n\right] \in$	✓	✓	✓	✓
$\mathcal{J} \in [\mathcal{I}]^\omega$, then $\bigcap \mathcal{I} \in$	✓	✓	✓	✓
$I, J \in \mathcal{I}$, J is a P-ideal, then $I \vee J \in$		✓		✓
$\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, then $\bigvee\{I_n : n \in \omega\} \in$				✓
$\langle I_n \rangle_{n\in\omega}$ is an increasing sequence of ideals from $\mathcal{I}$, then $\bigvee\{I_n : n \in \omega\} \in$		✓		✓
$\langle I_n \rangle_{n\in\omega}$ is an increasing sequence of analytic ideals from $\mathcal{I}$, then $\bigvee\{I_n : n \in \omega\} \in$		✓	✓	✓
$\langle I_n \rangle_{n\in\omega}$ is an increasing sequence of Borel ideals from $\mathcal{I}$, then $\bigvee\{I_n : n \in \omega\} \in$	✓	✓	✓	✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, $b\left[I\text{-}\prod_{n\in\omega} I_n\right] \in$		✓		✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega}$ is a sequence of analytic ideals from $\mathcal{I}$, $b\left[I\text{-}\prod_{n\in\omega} I_n\right] \in$	✓	✓	✓	✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, $I\text{-}\underline{\lim}_{n\in\omega} I_n \in$		✓		✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega}$ is a sequence of analytic ideals from $\mathcal{I}$, $I\text{-}\underline{\lim}_{n\in\omega} I_n \in$	✓	✓	✓	✓

Open problems

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Are there any examples of ideals which prove that the classes of all ideals satisfying $M^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$, $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$, $M^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$, and $H^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$ are pairwise distinct?

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Problem

Is there an ideal I such that $\bar{H}^{\Leftarrow}\left(\mathcal{F}_{\to_I}, \xrightarrow{QN}_I\right)$ does not hold?

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