Let $\mathcal{I}$ be a $\sigma$-ideal in a Polish space $X$ possessing a Borel base and containing singletons. Let us recall some notions concerning ideals.

**Definition 1.** We call a set $L \subseteq X$ an $\mathcal{I}$-Luzin set if $|L \cap A| < |L|$ for all $A \in \mathcal{I}$. $L$ is called a super $\mathcal{I}$-Luzin set, if $L \cap B$ is an $\mathcal{I}$-Luzin set for every $\mathcal{I}$-positive Borel set $B$.

In addition, let $\mathcal{I}$ be invariant $\sigma$-ideal in a Polish group $(X, +)$.

**Definition 2.** We say that $\mathcal{I}$ has a Weaker Smital Property (WSP), if there exists a countable dense set $D$ such that for every $\mathcal{I}$-positive Borel set $B$ a set $D + B$ is $\mathcal{I}$-residual.

In [2] and [1] we have explored how possessing WSP and or some other properties by $\mathcal{I}$ affects the behavior of $\mathcal{I}$-Luzin sets. Namely

**Theorem 1.** The following are true

(i) If $\mathcal{I}$ has WSP or $\mathcal{I}$ is $\kappa$-cc, then $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable.

(ii) $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable if, and only if, $\mathcal{I}$ is tall.

(iii) If $\mathcal{I}$-Luzin sets exist and $\mathcal{I}$ has WSP, then super $\mathcal{I}$-Luzin sets exist.

We have obtained the following results related to WSP.

**Theorem 2.** If $\mathcal{I}$ has WSP, then $\mathcal{I}$ is ccc or $\text{cov}(\mathcal{I}) = \omega_1$.

**Theorem 3.** If $\mathcal{I}$ has WSP, then the following are equivalent:

(i) $\text{cov}(\mathcal{I}) \geq \omega_2$;

(ii) For every family of sets $\{B_\alpha : \alpha < \omega_1\} \subseteq \text{Bor}(X) \setminus \mathcal{I}$ there is a set $T \subseteq \omega_1$ of cardinality $\omega_1$ such that $\bigcap_{\alpha \in T} B_\alpha \neq \emptyset$.

The latter theorem generalizes a result obtained in [3], which was the answer to a question asked by Peter Simon during 8th Winter School in Abstract Analysis.

The result concerning WSP and (non)measurability of $\mathcal{I}$-Luzin sets in the Theorem 1 is based on the following lemma.

**Lemma 1.** Let $P \subseteq \mathbb{R}$ be a perfect set. Then there exists a perfect set $P' \subseteq P$ such that $|P' \cap (x + P')| \leq 1$ for every $x \neq 0$.

It can be used to give a simple example of set that witnesses that a $\sigma$-ideal of countable sets $[\mathbb{R}]^{\leq \omega}$ does not have the Fubini Property. Let us recall that a pair $(\mathcal{I}, \mathcal{J})$ of $\sigma$-ideals $\mathcal{I} \subseteq P(X)$ and $\mathcal{J} \subseteq P(Y)$ has the Fubini Property, if

$$(\forall B \in \text{Bor}(X \times Y))(\{x \in X : B_x \notin \mathcal{J}\} \in \mathcal{I} \Rightarrow \{y \in X : B^y \notin \mathcal{I}\} \in \mathcal{J})$$

We will prove the following generalization of the above lemma.
Lemma 2. Let $T \subseteq \mathbb{Z}^\omega$ be a Laver (resp. Miller) tree. Then there exists a Laver (resp. Miller) subtree $T' \subseteq T$ such that

$$|[T'] \cap (x + [T'])| \leq \omega,$$

for all $x \in \mathbb{Z}^\omega$ satisfying $(\forall^\infty n)(x(n) \neq 0)$.

Let us denote by $\mathcal{L}$ the Laver ideal - a $\sigma$-ideal of sets that are not strongly dominating. We will use this lemma to prove the following theorem.

Theorem 4. Let $[\mathbb{Z}^\omega]^\omega \subseteq J \subseteq \mathcal{L}$. Then the pair $(\mathcal{L}, J)$ does not satisfy the Fubini Property.

These results were achieved together with Robert Rałowski and Szymon Żeberski.

References

