Fubini and Smital among trees

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Let $\mathcal{I}$ be a σ-ideal in a Polish space X possessing a Borel base and containing singletons. Let us recall some notions concerning ideals.

Definition 1. *We call a set $L \subseteq X$ an $\mathcal{I}$-Luzin set if $|L \cap A| < |L|$ for all $A \in \mathcal{I}$. L is called a super $\mathcal{I}$-Luzin set, if $L \cap B$ is an $\mathcal{I}$-Luzin set for every $\mathcal{I}$-positive Borel set B.*

In addition, let $\mathcal{I}$ be invariant σ-ideal in a Polish group $(X, +)$.

Definition 2. *We say that $\mathcal{I}$ has a Weaker Smital Property* (WSP)*, if there exists a countable dense set D such that for every $\mathcal{I}$-positive Borel set B a set $D + B$ is $\mathcal{I}$-residual.*

In [2] and [1] we have explored how possessing WSP and or some other properties by $\mathcal{I}$ affects the behavior of $\mathcal{I}$-Luzin sets. Namely

Theorem 1. *The following are true*

(i) If $\mathcal{I}$ has WSP *or $\mathcal{I}$ is κ-cc, then $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable.*

(ii) $\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable if, and only if, $\mathcal{I}$ is tall.

(iii) If $\mathcal{I}$-Luzin sets exist and $\mathcal{I}$ has WSP*, then super $\mathcal{I}$-Luzin sets exist.*

We have obtained the following results related to WSP.

Theorem 2. *If $\mathcal{I}$ has* WSP*, then $\mathcal{I}$ is ccc or* $\mathrm{cov}(\mathcal{I}) = \omega_1$.

Theorem 3. *If $\mathcal{I}$ has* WSP*, then the following are equivalent:*

(i) $\mathrm{cov}(\mathcal{I}) \geq \omega_2$*;*

(ii) For every family of sets $\{B_\alpha : \alpha < \omega_1\} \subseteq Bor(X) \backslash \mathcal{I}$ there is a set $T \subseteq \omega_1$ of cardinality ω_1 such that $\bigcap_{\alpha \in T} B_\alpha \neq \emptyset$.

The latter theorem generalizes a result obtained in [3], which was the answer to a question asked by Peter Simon during 8^{th} Winter School in Abstract Analysis.

The result concerning WSP and (non)measurability of $\mathcal{I}$-Luzin sets in the Theorem 1 is based on the following lemma.

Lemma 1. *Let $P \subseteq \mathbb{R}$ be a perfect set. Then there exists a perfect set $P' \subseteq P$ such that $|P' \cap (x + P')| \leq 1$ for every $x \neq 0$.*

It can be used to give a simple example of set that witnesses that a σ-ideal of countable sets $[\mathbb{R}]^{\leq \omega}$ does not have the Fubini Property. Let us recall that a pair $(\mathcal{I}, \mathcal{J})$ of σ-ideals $\mathcal{I} \subseteq P(X)$ and $\mathcal{J} \subseteq P(Y)$ has the Fubini Property, if

$$(\forall B \in Bor(X \times Y))(\{x \in X : B_x \notin \mathcal{J}\} \in \mathcal{I} \Rightarrow \{y \in X : B^y \notin \mathcal{I}\} \in \mathcal{J})$$

We will prove the following generalization of the above lemma.

Lemma 2. *Let $T \subseteq \mathbb{Z}^{<\omega}$ be a Laver (resp. Miller) tree. Then there exists a Laver (resp. Miller) subtree $T' \subseteq T$ such that*

$$|[T'] \cap (x + [T'])| \leq \omega,$$

for all $x \in \mathbb{Z}^\omega$ satisfying $(\forall^\infty n)(x(n) \neq 0)$.

Let us denote by $\mathcal{L}$ the Laver ideal - a σ-ideal of sets that are not strongly dominating. We will use this lemma to prove the following theorem.

Theorem 4. *Let $[\mathbb{Z}^\omega]^{\leq\omega} \subseteq \mathcal{J} \subseteq \mathcal{L}$. Then the pair $(\mathcal{L}, \mathcal{J})$ does not satisfy the Fubini Property.*

These results were achieved together with Robert Rałowski and Szymon Żeberski.

References

[1] Michalski M., On some relations between properties of invariant σ-ideals in Polish spaces, 14^{th} Students' Science Conference (2016), Fundamental research questions, pp. 29-33.

[2] Michalski M., Żeberski Sz., Some properties of $\mathcal{I}$-Luzin, Topology and its Applications, 189 (2015), pp. 122-135.

[3] Cichoń J., Szymański A., Weglorz B., On intersection of sets of positive Lebesgue measure, Colloquium Mathematicum, vol. 52, no. 2 (1987), pp. 173-174.