Smital and Fubini among trees

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Winter School in Abstract Analysis 2019,
section Set Theory and Topology
28.01.2019, Hejnice

Let $\mathcal{I}$ be a σ-ideal in a Polish space X possessing Borel base and containing singletons.

Definition

We call a set L $\mathcal{I}$-Luzin set if $|L \cap A| < |L|$ for every set $A \in \mathcal{I}$.
L is a super $\mathcal{I}$-Luzin set, if $L \cap B$ is an $\mathcal{I}$-Luzin set for each $B \in Bor(X) \setminus \mathcal{I}$.

Let $\mathcal{I}$ be a σ-ideal in a Polish group $(X, +)$.

Definition

We say that $\mathcal{I}$ has a Weaker Smital Property (WSP), *if there exists a countable dense set D such that for every Borel $\mathcal{I}$-positive set B a set $D + B$ is $\mathcal{I}$-residual.*

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Why Weaker? See:

A. Bartoszewicz, M. Filipczak, T. Natkaniec, On Smital properties, Topology and its Applications 158 (2011), pp. 2066-2075.

Definition

$\mathcal{I}$ is tall if every $\mathcal{I}$-positive Borel set contains a perfect set from $\mathcal{I}$.

Theorem

Let $\mathcal{I} \subseteq P(\mathbb{R})$ has WSP. *Then the following are true*

- *$\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable.*
- *If $\mathcal{I}$-Luzin sets exist then also super $\mathcal{I}$-Luzin sets exist.*

Theorem

$\mathcal{I}$-Luzin sets are $\mathcal{I}$-nonmeasurable, if, and only if, $\mathcal{I}$ is tall.

Question

Does WSP *imply* ccc*?*

Theorem

If $\mathcal{I}$ *has the* WSP, *then* $\mathcal{I}$ *is* ccc *or* $\mathrm{cov}(\mathcal{I}) = \omega_1$

Definition

Let $\mathcal{I} \subseteq P(X)$ and $\mathcal{J} \subseteq P(Y)$ be σ-ideals. We define a Fubini product $\mathcal{I} \otimes \mathcal{J}$ of these ideals as follows:

$$A \in \mathcal{I} \otimes \mathcal{J} \Leftrightarrow (\exists B \in Bor(X \times Y))(A \subseteq B \;\&\; \{x \in X : B_x \notin \mathcal{J}\} \in \mathcal{I})$$

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Proposition

If $\mathcal{I} \otimes \mathcal{J}$ is ccc then $\mathcal{I}$ and $\mathcal{J}$ are ccc too.

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We say that a σ-ideal $\mathcal{J} \subseteq P(Y)$ is Borel-on-Borel if for every Borel set $B \in Bor(X \times Y)$ a set $\{x \in X : B_x \in \mathcal{J}\}$ is Borel.

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If $\mathcal{I}$ *has the* WSP, *then* $\mathcal{I}$ *is* ccc *or* $\mathrm{cov}(\mathcal{I}) = \omega_1$

Proof.

Let $\{B_\alpha : \alpha < \omega_1\} \subseteq Bor(X \times Y) \setminus \mathcal{I} \otimes \mathcal{J}$ be a family of pairwise disjoint sets, where $\mathcal{J}$ is *Borel*-on-*Borel* and ccc. Let D witness WSP for $\mathcal{I}$.

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If $\mathcal{I}$ *has the* WSP, *then* $\mathcal{I}$ *is* ccc *or* $\text{cov}(\mathcal{I}) = \omega_1$

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- Let $A_\alpha = \{x \in X : B_x \notin \mathcal{J}\}$ for every $\alpha < \omega_1$.

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- Let $A_\alpha = \{x \in X : B_x \notin \mathcal{J}\}$ for every $\alpha < \omega_1$.
- If $\bigcap_{\alpha<\omega_1}(D + A_\alpha) = \emptyset$, then $cov(\mathcal{I}) = \omega_1$.

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- If $\bigcap_{\alpha<\omega_1}(D + A_\alpha) = \emptyset$, then $cov(\mathcal{I}) = \omega_1$.
- If not, then let $x \in \bigcap_{\alpha<\omega_1}(D + A_\alpha)$.
- Since D is countable, there is $d \in D$ and an uncountable set $T \subseteq \omega_1$ such that $x - d \in \bigcap_{\alpha \in T}(A_\alpha)$.

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- If $\bigcap_{\alpha<\omega_1}(D + A_\alpha) = \emptyset$, then $cov(\mathcal{I}) = \omega_1$.
- If not, then let $x \in \bigcap_{\alpha<\omega_1}(D + A_\alpha)$.
- Since D is countable, there is $d \in D$ and an uncountable set $T \subseteq \omega_1$ such that $x - d \in \bigcap_{\alpha \in T}(A_\alpha)$.
- A family $\{(B_\alpha)_{x-d} : \alpha \in T\}$ witnesses that $\mathcal{J}$ is not ccc, a contradiction, hence $\mathcal{I} \otimes \mathcal{J}$ is ccc, so $\mathcal{I}$ is ccc too.

□

Remark

Let $\mathcal{I}$ possess WSP. *The following are equivalent:*

- $\text{cov}(\mathcal{I}) \geq \omega_2$.
- *For every family of sets* $\{B_\alpha : \alpha < \omega_1\} \subseteq Bor(X) \backslash \mathcal{I}$ *there is a set* $T \subseteq \omega_1$ *of cardinality* ω_1 *such that* $\bigcap_{\alpha \in T} B_\alpha \neq \emptyset$.

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J. Cichoń, A. Szymański, B. Weglorz, On intersection of sets of positive Lebesgue measure, Colloquium Mathematicum, vol. 52, no. 2 (1987), pp. 173-174.

Theorem

If $\mathcal{I}$ has the WSP, *then $\mathcal{I}$ is tall.*

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Lemma

Let P be a perfect nonempty subset of $\mathbb{R}$. Then there exists a perfect (and nonempty!) set $P' \subseteq P$ for which $|P' \cap (x + P')| \leq 1$ for $x \neq 0$.

Definition

We say that a pair $(\mathcal{I}, \mathcal{J})$ *of* σ*-ideals has a Fubini Property* (FP), *if*

$$(\forall B \in Bor(X \times Y))(\{x \in X : B_x \notin \mathcal{J}\} \in \mathcal{I} \Rightarrow)\{y \in X : B^y \notin \mathcal{I}\} \in \mathcal{J})$$

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Example

$([\mathbb{R}]^{\leq\omega}, [\mathbb{R}]^{\leq\omega})$ does not have the FP.

Proof.

Let P be as in the Lemma. Consider

$$B = \{(x, y) : y \in P \wedge x \in P - y\}.$$

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Proposition

If $\mathcal{I}$ *is not tall, then* $(\mathcal{I}, [\mathbb{R}]^{\leq\omega})$ *does not have the* FP.

Definition

Let $T \subseteq \omega^{<\omega}$ *be a tree. Then*

- *for each* $\tau \in T$ $\mathrm{succ}(\tau) = \{n \in \omega : \tau^\frown n \in T\}$*;*
- $\mathrm{split}(T) = \{\tau \in T : |\mathrm{succ}(\tau)| \geq 2\}$*;*
- ω-$\mathrm{split}(T) = \{\tau \in T : |\mathrm{succ}(\tau)| = \aleph_0\}$*.*
- $\mathrm{stem}(T) \in T$ *is a node* τ *such that for each* $\sigma \subsetneq \tau$ $|\mathrm{succ}(\sigma)| = 1$ *and* $|\mathrm{succ}(\tau)| > 1$*.*

Definition

A tree T on ω is called

- *a Sacks tree or perfect tree, if for each node $\sigma \in T$ there is $\tau \in T$ such that $\sigma \subseteq \tau$ and $|succ(\tau)| \geq 2$;*
- *a Miller tree or superperfect tree, if for each node $\sigma \in T$ exists $\tau \in T$ such that $\sigma \subseteq \tau$ and $|\mathrm{succ}(\tau)| = \aleph_0$;*
- *a Laver tree, if for each node $\tau \supseteq stem(T)$ we have $|\mathrm{succ}(\tau)| = \aleph_0$;*

Main Lemma

Let $T \subseteq \mathbb{Z}^{<\omega}$ be a Laver (resp. Miller) tree. Then there exists a Laver (resp. Miller) subtree $T' \subseteq T$ such that

$$|[T'] \cap (x + [T'])| \leq \omega,$$

for all $x \in \mathbb{Z}^{\omega}$ satisfying $(\forall^{\infty} n)(x(n) \neq 0)$.

Definition

Let X, d *be a metric space. We say that a set* A *has a Distinct Distances Property* (DDP)*, if for every* $\delta > 0$ *we have* $d(x, y) = \delta$ *for at most one pair* $(x, y) \in A^2$ *(up to a swap).*

Remark

If a set $A \subseteq \mathbb{R}$ *has the* DDP*, then* $|A \cap (x + A)| \leq 1$ *for* $x \neq 0$*.*

Definition

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Remark

If a set $A \subseteq \mathbb{R}$ has the DDP*, then $|A \cap (x + A)| \leq 1$ for $x \neq 0$.*

Proposition

Let $(A_n : n \in \omega)$ be a sequence of infinite subsets of ω. Then there exists a sequence $(B_n : n \in \omega)$ such that:

- *$B_n \subseteq A_n$ for all $n \in \omega$,*
- *$B_n \cap B_k = \emptyset$ for $n \neq k$,*
- *$\bigcup_{n \in \omega} B_n$ satisfies* DDP.

Let $T \subseteq \mathbb{Z}^{<\omega}$ be a tree and let us fix $x \in \mathbb{Z}^\omega$. Let us denote

$$x + T = \{x \restriction |\sigma| + \sigma : \sigma \in T\}.$$

Let us observe that

Remark

For $\sigma \in T$ and $x \in \mathbb{Z}^\omega$ we have:

$$\text{succ}_{x+T}(x \restriction \sigma + \sigma) = \text{succ}_T(\sigma) + x(|\sigma|)$$

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Proof.

Let $A_\sigma = \mathrm{succ}_T(\sigma)$ for all $\sigma \in T$ and apply the Proposition to obtain a family $\{B_\sigma : \sigma \in T\}$. Set T' such that $\mathrm{succ}_{T'}(\sigma) = B_\sigma$.

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Let us denote the Laver ideal by $\mathcal{L}$.

Theorem

Let $[\mathbb{Z}^\omega]^{\leq\omega} \subseteq \mathcal{J} \subseteq \mathcal{L}$. Then the pair $(\mathcal{L}, \mathcal{J})$ does not satisfy the Fubini Property.

Thank you for your attention!