Superperfect Mycielski
(Shame of Szymon)

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Mycielski theorem

- Let $A \subseteq [0,1] \times [0,1]$ be comeager.
  Then there exists a perfect set P such that
  $$P \times P \subseteq A \cup \Delta.$$
- Assume that $A \subseteq [0,1] \times [0,1]$ has measure 1.
  Then there exists a perfect set P such that
  $$P \times P \subseteq A \cup \Delta.$$

Definition

T a tree $T \subseteq \omega^{<\omega}$.

$$[T] = \{x \in \omega^\omega : \ \forall n \, x \restriction n \in T\}$$

1. T is perfect if
   $(\forall \sigma \in T)(\exists \tau \in T)(\sigma \subseteq \tau \wedge (\exists n \neq m)(\tau^\frown n, \tau^\frown m \in T)$
2. T is superperfect if
   $(\forall \sigma \in T)(\exists \tau \in T)(\sigma \subseteq \tau \wedge (\exists^\infty n)(\tau^\frown n \in T)$

Theorem 1 (Category case)

For every comeager set G of $\omega^\omega \times \omega^\omega$ there exists a superperfect set $M \subseteq \omega^\omega$ and a perfect set $P \subseteq M$ such that $P \times M \setminus \Delta \subseteq G$.

Lemma 1

For every open dense set $U \subseteq X^2$ and two open sets $V_1, V_2 \subseteq X$ there are a basic open sets $B_1 \subseteq V_1$ and $B_2 \subseteq V_2$ such that $B_1 \times B_2 \subseteq U$ and $B_2 \times B_1 \subseteq U$.

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Lemma 2

For every open dense set $U \subseteq X^2$ and a finite sequence of open sets $(V_k : k < n)$ in X there is a sequence of basic open sets $(B_k : k < n)$ such that $B_k \subseteq V_k$ and for distinct $k, l \in \omega$ we have $B_k \times B_l \subseteq U$.

Proof of Theorem 1.

$G = \bigcap_{n\in\omega} U_n$ where $(U_n)_{n\in\omega}$ Define recursively a sequence $(B_n : n \in \omega)$ of sets such that for each $n \in \omega$ the set $B_n = \{\tau_\sigma : \sigma \in n^{\leq n}\}$ consists of nodes satisfying:

1. $\tau_\emptyset = \emptyset$ and $\tau_{\sigma_1} \subseteq \tau_{\sigma_2}$ for $\sigma_1 \subseteq \sigma_2$;
2. $\tau_{\sigma^\frown k} \cap \tau_{\sigma^\frown j} = \tau_\sigma$ for $\sigma \in n^{<n}$ and distinct $k, j < n$;
3. a sequence of clopen sets $([\tau_\sigma] : \sigma \in n^n)$ satisfies the thesis of Lemma 2 for U_n.

$$T = \{\tau \in \omega^{<\omega} : (\exists \tau' \in \bigcup_{n\in\omega} B_n)(\tau \subseteq \tau')\}$$

$$T_2 = \{\tau \in \omega^{<\omega} : (\exists \sigma \in 2^n)(\tau \subseteq \tau_\sigma)\}$$

Then

$$[T] \times [T_2] \subseteq G \setminus \Delta.$$

□

Definition

Let T be a tree on a set A. Then

- for each $t \in T$ $succ(t) = \{a \in A : t^\frown a \in T\}$;
- $split(T) = \{t \in T : |succ(t)| \geq 2\}$;
- $\omega\text{-}split(T) = \{t \in T : |succ(t)| = \aleph_0\}$;
- for $s \in T$ $Succ_T(s) = \{t \in split(T) : s \subseteq t, (\forall t' \in T)(s \subseteq t' \subseteq t \rightarrow t' \notin split(T))\}$;
- for $s \in T$ $\omega\text{-}Succ_T(s) = \{t \in \omega\text{-}split(T) : s \subseteq t, (\forall t' \in T)(s \subseteq t' \subseteq t \rightarrow t' \notin \omega\text{-}split(T))\}$;
- $stem(T) \in T$ is a node τ such that for each $s \subsetneq \tau$ $|succ(s)| = 1$ and $|succ(\tau)| > 1$.

Definition

A tree T on ω is called

- Sacks tree or perfect tree, if for each node $s \in T$ there is $t \in T$ such that $s \subseteq t$ and $|succ(t)| \geq 2$;
- Miller tree or superperfect tree, if for each node $s \in T$ exists $t \in T$ such that $s \subseteq t$ and $|succ(t)| = \aleph_0$;
- Laver tree, if for each node $t \supseteq stem(T)$ we have $|succ(t)| = \aleph_0$;
- Hechler tree, if for each node $t \supseteq stem(T)$ we have that a set $\{n \in \omega : t^\frown n \notin T\}$ is finite;

Lemma 3

There exists a dense G_δ set $G \subseteq \omega^\omega$ which contains no body of any Laver tree.

Proof.

$G = \{x \in \omega^\omega : \exists^\infty n\, x(n) = 0\}$ □

Corollary

Mycielski Theorem for the category does not hold in the case of Laver trees.

Proof.

Let us take G as in the Lemma 3. Set $G' = G \times \omega^\omega$. □

We will work in $[0,1]^2$ and we will recognize superperfect sets as homeomorphic images of bodies of Miller trees from ω^ω in $[0,1] \setminus \mathbb{Q}$.

Theorem 2 (Measure case)

For every measure 1 set F of $\omega^\omega \times \omega^\omega$ there exists a superperfect set $M \subseteq \omega^\omega$ and a perfect set $P \subseteq M$ such that $P \times M \setminus \Delta \subseteq F$.

Lemma 4

For every F_σ set F there is an F_σ set $\widetilde{F} \subseteq F$ of the same measure such that $\widetilde{F}^* \subseteq F$.

F^* denotes points of density 1

Proof.

Let $F = \bigcup_{n\in\omega} F_n$, where $(F_n)_{n\in\omega}$ is an ascending sequence of closed sets. $\lambda(F^* \setminus F) = 0$, thus for every $n \in \omega$ let U_n be an open set of measure $< \frac{1}{n+1}$ such that $F^* \setminus F \subseteq U_n$. For every $n \in \omega$ let us set

$$\widetilde{F}_n = F_n \setminus U_n$$

and

$$\widetilde{F} = \bigcup_{n\in\omega} \widetilde{F}_n.$$

□

Lemma 5

Let $\varepsilon > 0$, $F \subseteq [0,1]^2$ be an F_σ set of full measure and $(U_k : k < n)$ a finite sequence of open subsets of $[0,1]$. Then there exists a sequence of open intervals with rational endpoints $(I_k : k < n)$ such that for distinct $i, j < n$ we have $\lambda(I_k \times I_j \cap F) > 1 - \varepsilon$.

Fact

Mycielski Theorem for the measure does not hold in the case of Laver trees.

Proof.

For every $n \in \omega$ let H_n be a Hechler tree such that for every $\sigma \in \omega^{\leq n}$ we have $split_{H_n}(\sigma) = \omega$ and for $\sigma \in \omega^{>n}$ it is true that $split_{H_n}(\sigma) \neq \omega$ (but still cofinite). Let us set then $G = \bigcup_{n\in\omega}[H_n]^c$. □

Thank you for your attention!