Order dimension of Turing degrees

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Order dimension

Definition

Suppose $(\mathbb{P}, \leq)$ is a poset. The **order dimension** of $\mathbb{P}$ is the least κ such that there exists $\langle \leq_i : i < \kappa \rangle$ where each $\leq_i$ is a linear order on $\mathbb{P}$ that extends $\leq$ and for every $a \neq b$ in $\mathbb{P}$, if $a \not\leq b$, then for some $i < \kappa$, $b \leq_i a$.

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Higuchi-Lempp-Raghavan-Stephan showed that it is $\leq \kappa$ if $\mathfrak{c} = \kappa^+$ and κ has uncountable cofinality and asked if it could be equal to $\mathfrak{c}$ when CH fails?

What's inside $\mathcal{D}$?

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Question (Sacks)

Must every locally countable poset of size continuum embed in $\mathcal{D}$?

A poset inside $\mathcal{D}$

Definition

Let $\mathbb{H}_\kappa$ be the poset consisting of $\kappa \sqcup [\kappa]^{\aleph_0}$ with the ordering $a < B$ iff $a \in \kappa$, $B \in [\kappa]^{\aleph_0}$ and $a \in B$.

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Proof: We'd like to start with a Turing independent perfect set of reals C (the join of any finite $F \subseteq C$ does not compute another real in $C \setminus F$) and try to add countable joins y_A for $A \in [C]^{\aleph_0}$. Note that y_A's should be pairwise Turing incomparable and no y_A computes a member of $C \setminus A$.

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We use the following perfect set version of the exact pair theorem of Spector.

Lemma (Spector)

Suppose $\langle a_n : n < \omega \rangle$ is $\leq_T$-increasing. Put $\mathcal{I} = \{z : (\exists n)(z \leq_T a_n)\}$. Then there is a perfect set $P \subseteq 2^\omega$ such that for any $x \neq y$ in P, x, y form an **exact pair** for $\mathcal{I}$, i.e., $\{z : z \leq_T x \wedge z \leq_T y\} = \mathcal{I}$.

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Let $\{c_\xi : \xi < \mathfrak{c}\}$ be a Turing independent set of reals. For each $A \in [\mathfrak{c}]^{\aleph_0}$ fix a perfect set P_A of exact pairs for the Turing ideal generated by A. Inductively choose $X \in [\mathfrak{c}]^\mathfrak{c}$ and $y_A \in P_A$ for $A \in [X]^{\aleph_0}$ such that y_A's are pairwise Turing incomparable and y_A computes c_ξ for $\xi \in X$ iff $\xi \in A$. □

The order dimension of $\mathbb{H}_\kappa$

Definition

Suppose κ is uncountable and $\theta < \kappa$. $\star(\theta, \kappa)$ is the following statement: For every $\langle <_i : i < \theta \rangle$ where each $<_i$ is a linear order on κ, there exist X, α such that $X \in [\kappa]^{\aleph_0}$, $\alpha \in \kappa \setminus X$ and for every $i < \theta$, there exists $\beta \in X$ such that $\alpha \leq_i \beta$.

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Proof: Suppose the order dimension of $\mathbb{H}_\kappa \leq \theta$. Let $\langle (\kappa \sqcup [\kappa]^{\aleph_0}, <_i) : i < \theta \rangle$ witness this. Suppose $A \in [\kappa]^{\aleph_0}$ and $\alpha \in \kappa \setminus A$. Since $\alpha \notin A$, for some $i < \theta$, $A <_i \alpha$. Hence for every $\beta \in A$, $\beta <_i A <_i \alpha$. It follows that $\langle <_i \restriction \kappa : i < \theta \rangle$ witnesses the failure of $\star(\theta, \kappa)$. □

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Exercise

Prove the converse.

Strongly saturated ideals

Definition

Suppose κ is regular uncountable and $\mathcal{I}$ is a κ-additive ideal on κ that contains every bounded subset of κ. Call $\mathcal{I}$ **strongly saturated** if for every family $\mathcal{F}$ of $\mathcal{I}$-positive sets, if $|\mathcal{F}| < \kappa$, then there is a countable set that meets every member of $\mathcal{F}$.

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If $\mathcal{I}$ is the null ideal of a witnessing normal measure on a real valued measurable cardinal κ, this is Problem EG(h) on Fremlin's list.

Strongly saturated ideals and order dimension

Lemma

Let κ be regular uncountable and $\theta < \kappa$. Suppose there is a κ-additive ideal $\mathcal{I}$ on κ (containing all bounded subset of κ) such that for every $\mathcal{A} \in [\mathcal{I}^+]^\theta$, there exists $X \in [\kappa]^{\aleph_0}$ such that for every $A \in \mathcal{A}$, $X \cap A \neq \emptyset$. Then $\star(\theta, \kappa)$ holds. Hence the order dimension of $\mathbb{H}_\kappa$ is at least θ.

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Proof: Let $\{<_i : i < \theta\}$ be a family of linear orders on κ. For each $i < \theta$, let $R_i = \{\alpha < \kappa : \{\beta < \kappa : \beta \geq_i \alpha\} \in \mathcal{I}\}$. Let $\Gamma = \{i < \theta : R_i \in \mathcal{I}^+\}$. Choose $A_0 \in [\kappa]^{\aleph_0}$ such that $A_0 \cap R_i$ is infinite for each $i \in \Gamma$. Let $W = \kappa \setminus \bigcup_{i \in \Gamma} \{\beta < \kappa : (\exists \alpha \in A_0 \cap R_i)(\beta \geq_i \alpha)\}$. Note that $\kappa \setminus W \in \mathcal{I}$. Choose $\alpha \in W \setminus \bigcup_{i \in \theta \setminus \Gamma} R_i$. Choose $A_1 \in [\kappa]^{\aleph_0}$ such that it meets $\{\beta < \kappa : \beta \geq_i \alpha\}$ at an infinite set for every $i \in \kappa \setminus \Gamma$. Let $A = (A_0 \cup A_1) \setminus \{\alpha\}$. Then α is not $\leq_i$ above A for any $i < \theta$. □

The model

Theorem

Suppose κ is measurable with a witnessing normal prime ideal $\mathcal{I}$. Let $\mathbb{C}$ be the forcing for adding κ Cohen reals. Then in $V^{\mathbb{C}}$, $\mathcal{J} = \{A \subseteq \kappa : (\exists B \in \mathcal{I})(A \subseteq B)\}$ is a strongly saturated ideal on κ. So the order dimension of $\mathbb{H}_{\mathfrak{c}}$ is $\mathfrak{c}$ in this model.

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Proof: It suffices to show the following.

Claim

Suppose $p \in \mathbb{C}$, $\theta < \kappa$ and $\mathring{A}_i \in V^{\mathbb{C}} \cap \mathcal{P}(\kappa)$ for $i < \theta$ are such that $p \Vdash \mathring{A}_i \in \mathcal{J}^+$ for every $i < \theta$. Then there are $q \leq p$ and $X \in [\kappa]^{\aleph_0}$ such that $q \Vdash (\forall i < \theta)(\mathring{A}_i \cap X \neq \emptyset)$.

The model

Proof: Let m be a normal witnessing measure on κ and $\mathcal{I}$ be its null ideal. WLOG, $p = 1_{\mathbb{C}}$. Let $p_{i,\alpha} = [[\alpha \in \mathring{A}_i]]_{\mathbb{C}}$. Note that for every $X \subseteq \kappa$, if $m(X) = 1$, then $\bigcup_{\alpha \in X} p_{i,\alpha} = 1_{\mathbb{C}}$. Choose $X \subseteq \kappa$ such that $m(X) = 1$ and for every $\alpha \in X$ and $i < \theta$, $p_{i,\alpha} \neq 0_{\mathbb{C}}$. Let $\text{supp}(p_{i,\alpha}) = S_{i,\alpha} \in [\kappa]^{\aleph_0}$. Using normality of the null ideal of m, for each $i < \theta$, choose $Y_i \subseteq X$ such that $m(Y_i) = 1$ and for every $\alpha, \beta \in Y_i$

- $S_{i,\alpha}$ and $S_{i,\beta}$ have the same order type,
- $S_{i,\alpha} \cap \alpha = W_i$ does not depend on α and
- $(2^{S_{i,\alpha}}, p_{i,\alpha}) \cong (2^{S_{i,\beta}}, p_{i,\beta})$ which means that the unique order preserving bijection from $S_{i,\alpha}$ to $S_{i,\beta}$ sends $p_{i,\alpha}$ to $p_{i,\beta}$.

The model

Let $Y = \bigcap_{i<\theta} Y_i$. Choose $X \in [Y]^{\aleph_0}$ such that for every $i < \theta$, $\{S_{i,\alpha} \setminus W_i : \alpha \in X\}$ are pairwise disjoint. It is not hard to check that for every $i < \theta$, $\bigcup_{\alpha \in X} p_{i,\alpha}$ is open dense in 2^κ. Hence $\Vdash (\forall i < \theta)(X \cap \mathring{A}_i \neq \emptyset)$. □

Thank You!