Hereditary (bi)coreflective subcategories in certain categories of semitopological groups

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- every subcategory of **STopGr** is isomorphism-closed

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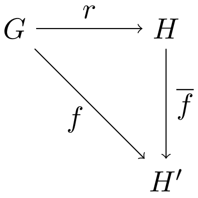

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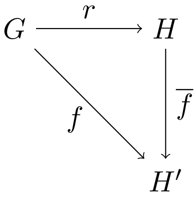

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$$\begin{array}{ccc} G & \xrightarrow{\quad r \quad} & H \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & H' \end{array}$$

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- extremal epireflective: every reflection is an extremal epimorphism (quotient map)

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  - abelian semitopological groups (**STopAb**)
  - torsion-free semitopological groups
  - Hausdorff semitopological groups

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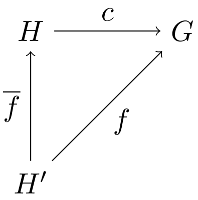

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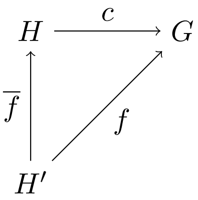

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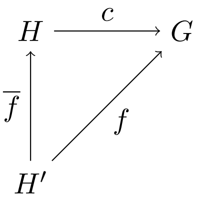

- monocoreflective: every coreflection is a monomorphism
- bicoreflective: every coreflection is a bimorphism (monomorphism and epimorphism)

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  e.g. **QTopGr** in **STopGr**, **TopGr** in **PTopGr**

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Thank you for your attention.