Some results on ultrapower capturing

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Large cardinals

A reasonable definition of large cardinals:

κ is large if there is an elementary embedding $j\colon V \to M$ where

- κ is the critical point (meaning that κ is the first ordinal moved by j), and
- M is a transitive inner model which is "close" to V.

Examples include:

- measurable cardinals: any M is fine;
- θ-strong cardinals: $V_\theta \subseteq M$;
- θ-supercompact cardinals: ${}^{\theta}M \subseteq M$;
- ...

Requiring $M = V$ is inconsistent (Kunen).

Measurable cardinals

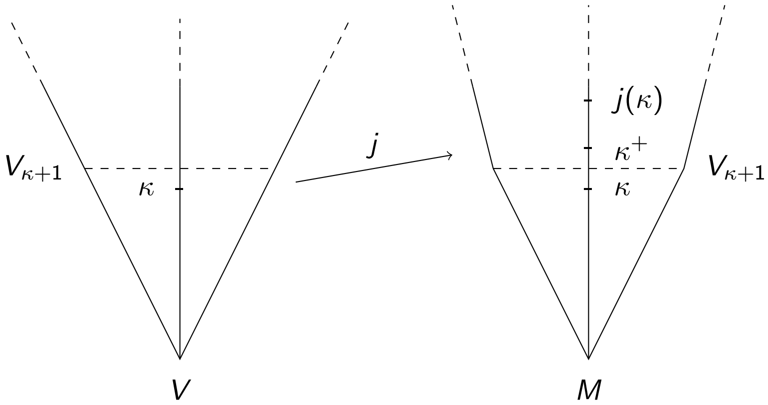
An alternative definition of measurability:

κ is measurable if there is a κ-complete nonprincipal ultrafilter on κ (called a measure).

If U is such an ultrafilter then the ultrapower construction gives the model $M = \mathrm{Ult}(V, U)$ and the embedding $j\colon V \to M$.

In particular, we get $\mathcal{P}(\kappa) \in M$, or equivalently $V_{\kappa+1} \in M$. On the other hand, it is an easy fact that $U \notin M$, which means $\mathcal{P}(\mathcal{P}(\kappa)) \notin M$.

Ultrapowers



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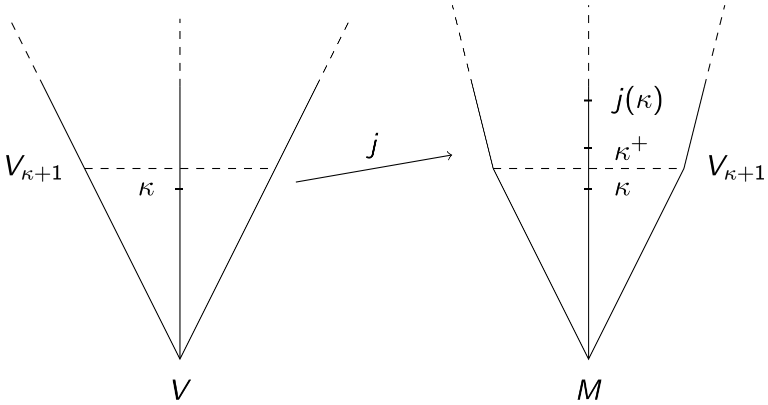


Question (Steel)

Is it consistent that κ carries a normal measure whose ultrapower contains all of $\mathcal{P}(\kappa^+)$?

The capturing property

Definition

If κ, λ are cardinals, say that $\mathrm{CP}(\kappa, \lambda)$ holds if there is a normal measure on κ whose ultrapower contains $\mathcal{P}(\lambda)$.

We observed earlier that $\mathrm{CP}(\kappa, \kappa)$ holds and $\mathrm{CP}(\kappa, 2^\kappa)$ fails. Steel asked about the consistency of $\mathrm{CP}(\kappa, \kappa^+)$.

Theorem (Cummings, 1993)

$\mathrm{CP}(\kappa, \kappa^+)$ *is consistent relative to a* $(\kappa + 2)$*-strong cardinal* κ*. Moreover, the hypothesis is optimal.*

The proof strategy

How does one produce such a measure/ultrapower?

1. Start with a sufficiently fat embedding $j\colon V \to M$ that captures $\mathcal{P}(\kappa^+)$ (but is not necessarily a measure ultrapower).
2. Force to $V[G]$ in which $2^\kappa = \kappa^{++}$ and extend j to $j^*\colon V[G] \to M[H]$.
3. Hope for the best?

What saves us is the following key fact:

Fact

If $j\colon V \to M$ is a nice elementary embedding with critical point κ that can be extended to a forcing extension $j^*\colon V[G] \to M[H]$ in which 2^κ is large enough, then j^* is the ultrapower embedding by a normal measure on κ.

Capturing at the least measurable

Cummings showed that a measurable cardinal satisfying $\mathrm{CP}(\kappa, \kappa^+)$ is large in an inner model. It is less clear whether the capturing property has any direct implications about the size of κ in V. In the previous proof κ started out quite large, and this remains true in the final model.

Theorem (H.–Honzík)

It is consistent relative to a $(\kappa + 2)$-strong cardinal κ that $\mathrm{CP}(\kappa, \kappa^+)$ holds at the least measurable cardinal κ.

The modified proof strategy

We start with a fat embedding $j\colon V \to M$ again, but this time the forcing has to destroy all the measurables below κ in addition to forcing $2^\kappa = \kappa^{++}$.

One could try to first force $\mathrm{CP}(\kappa, \kappa^+)$ and only later make κ the least measurable, but this is not likely to work.

The solution is to use a forcing that simultaneously kills measurability and adds subsets to the cardinal in question.

The Apter–Shelah forcing

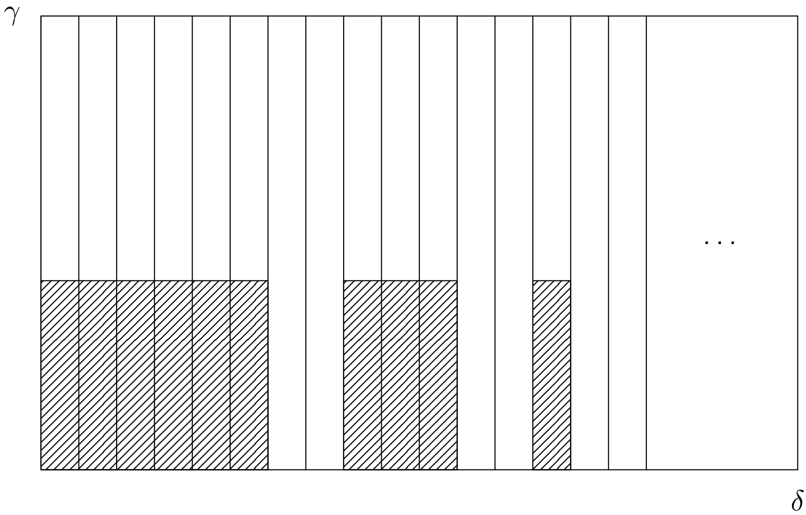
Let γ be inaccessible and $\delta > \gamma$ regular. Fix a nonreflecting stationary set $S \subseteq \delta \cap \mathrm{Cof}(\omega)$ and let $\vec{X} = \langle X_\alpha \,;\, \alpha \in S \rangle$ be an S-ladder system (meaning that each X_α is an ω-sequence cofinal in α).

The forcing $\mathbb{A}(\gamma, \delta, \vec{X})$ has conditions (p, Z), where

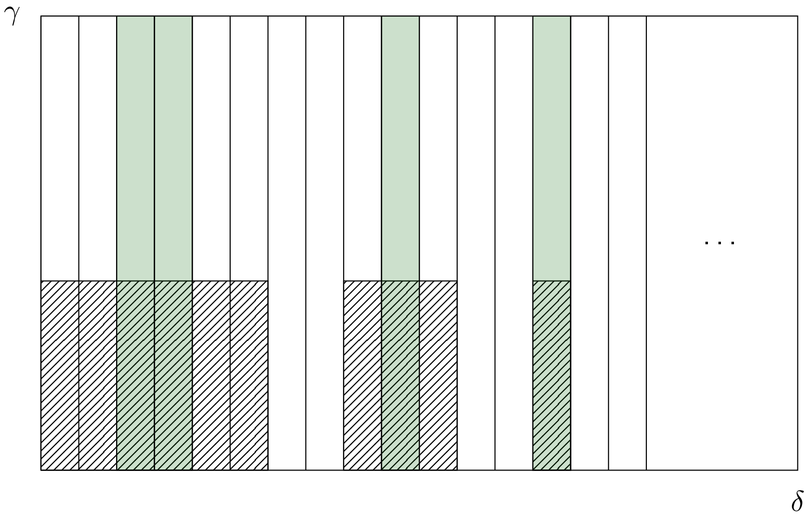
1. p is a uniform Cohen condition in $\mathrm{Add}(\gamma, \delta)$;
2. $Z \subseteq \vec{X}$;
3. $\forall X_\alpha \in Z \colon X_\alpha \subseteq \mathrm{supp}(p)$.

The side condition contains promises that the intersection of those countably many columns (or their complements) will never get a new element.

A condition



A condition



What is the forcing good for?

Unsurprising fact

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Very suprising fact

If C is a generic club disjoint from S, then $\mathbb{A}(\gamma, \delta, \vec{X})$ is, in $V[C]$, equivalent to $\mathrm{Add}(\gamma, \delta)$.

Outline of proof

Start with a $(\kappa+2)$-strong κ and a fat embedding $j\colon V \to M$.

Define a forcing iteration $\mathbb{P}_\kappa$ which forces at each inaccessible $\gamma < \kappa$ with

$$\mathbb{S}_{\gamma^{++}} * \mathbb{A}(\gamma, \gamma^{++}, \vec{X})$$

where $\mathbb{S}_{\gamma^{++}}$ adds a nonreflecting stationary set $\dot{S} \subseteq \gamma^{++} \cap \mathrm{Cof}(\omega)$ and $\vec{X}$ is a $\clubsuit_{\gamma^{++}}(\dot{S})$-sequence.

The actual forcing will be

$$\begin{aligned} \mathbb{P} &= \mathbb{P}_\kappa * \mathbb{S}_{\kappa^{++}} * (\mathbb{A}(\kappa, \kappa^{++}, \vec{Y}) \times \mathbb{C}(\dot{S})) \\ &\approx \mathbb{P}_\kappa * \mathrm{Add}(\kappa^{++}, 1) * \mathrm{Add}(\kappa, \kappa^{++}) \end{aligned}$$

Forcing with $\mathbb{P}$ kills all of the measurables below κ and one can show that j can be lifted to this forcing extension. The lifted embedding witnesses $\mathrm{CP}(\kappa, \kappa^+)$ in the extension.

Some related facts

One can play around with the values of 2^κ and capture powersets above κ^+ as well.

Theorem

If $\kappa < \lambda$ are cardinals and cf$(\lambda) > \kappa$, then $\mathrm{CP}(\kappa, < \lambda)$ is consistent relative to an H_λ-strong cardinal κ. In this model $2^\kappa = \lambda$.

One might also ask whether it is possible that κ carries very few measures but the capturing property nevertheless holds.

Theorem

It is consistent relative to a $(\kappa + 2)$-strong cardinal κ that κ carries a unique normal measure and that measure witnesses $\mathrm{CP}(\kappa, \kappa^+)$.

Question

In the last theorem, can κ be made to be the least measurable?

A local version of capturing

Definition

If κ, λ are cardinals, say that $\mathrm{LCP}(\kappa, \lambda)$ holds if there is, for each $x \subseteq \lambda$, a normal measure on κ whose ultrapower contains x.

The local version stretches a bit further than full capturing: by an old argument of Solovay, $\mathrm{LCP}(\kappa, 2^\kappa)$ holds at any 2^κ-supercompact or $(\kappa + 2)$-strong κ. It is not difficult to see that $\mathrm{LCP}(\kappa, (2^\kappa)^+)$ still fails.

Question

If κ is θ-supercompact for some $\kappa < \theta < 2^\kappa$, does $\mathrm{LCP}(\kappa, \theta)$ *hold?*

The consistency strength of local capturing

Fact

If LCP$(\kappa, 2^\kappa)$ *holds then* κ *has maximal Mitchell rank.*

We also get a bound from the other side.

Theorem

If $o(\kappa) \geq \kappa^{++}$ *then* LCP(κ, κ^+) *holds in Mitchell's model* $L[\vec{U}]$.

Actually, in this model there is a single function f such that $[f]_U$ can be any subset of κ^+ by a judicious choice of U.

Thank you.