Polarised Partition Relations for Order Types

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We call an order type φ *additively decomposable* if there are types ψ and τ such that $\varphi = \psi + \tau$ but neither $\varphi \leqslant \psi$ nor $\varphi \leqslant \tau$. We call it *unionwise decomposable* if there is an ordered set $\langle X, < \rangle$ of type φ and a $Y \subseteq X$ such that neither $\varphi \leqslant \mathrm{otp}(\langle Y, < \rangle)$ nor $\varphi \leqslant \mathrm{otp}(\langle X \setminus Y, < \rangle)$. We call it *multiplicatively decomposable* if there are types ψ and τ such that $\varphi = \psi\tau$ but neither $\varphi \leqslant \psi$ nor $\varphi \leqslant \tau$. We call it *typewise decomposable* if there is an ordered set $\langle X, <_X \rangle$ and for every $x \in X$ disjoint ordered sets $\langle Y_x, <_x \rangle$ such that the set $\langle \bigcup_{x \in X} Y_x, < \rangle$ has type φ if $a < b$ is given by $\exists x\, (\exists y \colon a \in x \wedge b \in y \wedge x <_X y) \vee (a \in x \wedge b \in x \wedge a <_x b)$ and furthermore neither $\varphi \leqslant \mathrm{otp}(\langle X, <_X \rangle)$ nor $\varphi \leqslant \mathrm{otp}(\langle Y_x, <_x \rangle)$ for any $x \in X$.

An order type is called (additively, unionwise, multiplicatively, typewise) *indecomposable* if it fails to be (additively, unionwise, multiplicatively, typewise) decomposable.

Observation

An ordinal is

- *. . . additively/unionwise indecomposable if and only if it is of the form* ω^α *for an ordinal* α,
- *. . . multiplicatively indecomposable if and only if it is of the form* ω^{ω^α} *for an ordinal* α,
- *. . . typewise indecomposable if and only if it is regular.*

Notation

$\eta := \mathrm{otp}(\mathbb{Q})$.

Definition

An order-type φ is called *scattered* if $\eta \nleq \varphi$.

Theorem ([Hausdorff, 1908, Satz XII])

The class of scattered order types is the smallest non-empty class containing all reversals and well-ordered sums.

Corollary

Up to equimorphism, the only countable typewise indecomposable order types are

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The class of scattered order types is the smallest non-empty class containing all reversals and well-ordered sums.

Corollary

Up to equimorphism, the only countable typewise indecomposable order types are 0, 1, 2, ω, ω^*, *and* η.

Notation (Erdős and Rado [1956])

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma & \varepsilon \\ \delta & \zeta \end{pmatrix}.$$

This relation states that for every colouring $\chi \colon A \times B \longrightarrow 2$ of a set A of size α and a set B of size β, either there is a $C \subseteq A$ of size γ and a $D \subseteq B$ of size δ such that $\chi\big[C \times D\big] = \{0\}$ or there is an $E \subseteq A$ of size ε and a $Z \subseteq D$ of size ζ such that $\chi\big[E \times Z\big] = \{1\}$.

Observation

If φ is a unionwise decomposable order type and ψ is any order type,

$$\text{then } \binom{\psi}{\varphi} \not\to \begin{pmatrix} 1 & 1 \\ \varphi & \varphi \end{pmatrix}.$$

Observation

$$\binom{\eta}{\eta} \not\to \begin{pmatrix} 1 & \aleph_0 \\ \aleph_0 & 1 \end{pmatrix}.$$

Observation

For all natural numbers m, n and all unionwise indecomposable types φ,

$$\begin{pmatrix} \varphi \\ mn+1 \end{pmatrix} \longrightarrow \begin{pmatrix} \varphi \\ n+1 \end{pmatrix}_m .$$

Proposition (Klausner and W.)

If k, m and n are natural numbers, then

$$\begin{pmatrix} \omega^k \\ \omega^m \end{pmatrix} \longrightarrow \begin{pmatrix} \omega^k & n \\ \omega^m & n \end{pmatrix} .$$

This can be proved using Ramsey's Theorem, a technique which was first used in Haddad and Sabbagh [1969] for the ordinary partition relation.

Lemma

For all order types ρ, τ, φ *and* ψ, $\rho \longrightarrow (2\tau, \varphi + \psi, \psi + \varphi)^2$ *implies*

$$\begin{pmatrix}\rho\\ \rho\end{pmatrix} \longrightarrow \begin{pmatrix}\tau & \varphi\\ \tau & \psi\end{pmatrix}.$$

Theorem ([Erdős and Rado, 1956, Theorem 6])

$\eta \longrightarrow (\eta, \aleph_0)^2$.

Theorem (Larson [1973–1974])

For all natural numbers n, $\omega^\omega \longrightarrow (\omega^\omega, n)^2$.

Proposition

For all natural numbers k,

$$\begin{pmatrix}\eta\\ \eta\end{pmatrix} \longrightarrow \begin{pmatrix}\eta & k\\ \eta & k\end{pmatrix}.$$

Proposition

For all natural numbers k,

$$\begin{pmatrix}\omega^\omega\\ \omega^\omega\end{pmatrix} \longrightarrow \begin{pmatrix}\omega^\omega & k\\ \omega^\omega & k\end{pmatrix}.$$

At this point we would like to recall the notion of *pinning*, cf. Galvin and Larson [1974/1975].

Definition

An order type φ can be *pinned* to an order type ψ (written as $\varphi \to \psi$) if for every ordered set F of type φ and P of type ψ there is a function (a so-called *pinning map*) $f : F \longrightarrow P$ such that every $f[X] \in [P]^{\psi}$ for every $X \in [F]^{\varphi}$.

Corollary

For all natural numbers k,

$$\binom{\eta}{\omega} \longrightarrow \binom{\eta \; k}{\omega \; k} \text{ and } \binom{\omega^{\omega}}{\omega} \longrightarrow \binom{\omega^{\omega} \; k}{\omega \;\; k}.$$

Lemma

For all natural numbers k and m and all order types φ and ψ and collections of order types $\langle \sigma_i \mid i < k \rangle$ and $\langle \tau_j \mid j < m \rangle$, if

$$\begin{pmatrix} \sigma_i \\ \tau_j \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_i \, \varphi \\ \tau_j \, \psi \end{pmatrix}$$

for all $i < k$ and all $j < m$, then

$$\begin{pmatrix} \sum_{i<k} \sigma_i \\ \sum_{j<m} \tau_j \end{pmatrix} \longrightarrow \begin{pmatrix} \sum_{i<k} \sigma_i \, \varphi \\ \sum_{j<m} \tau_j \, \psi \end{pmatrix}.$$

Theorem

For all ordinals $\alpha, \beta < \omega^\omega$ and all natural numbers n,

$$\binom{\omega\alpha}{\omega\beta} \longrightarrow \binom{\omega\alpha\ n}{\omega\beta\ n}.$$

Definition (van Douwen [1984])

A *tower* is a sequence $\langle x_\xi \mid \xi < \alpha \rangle$ of infinite sets of natural numbers such that for $\gamma < \beta$, the set x_γ almost contains x_β. A tower is *extendible* if there is an infinite set almost contained in every member of it. The *tower number* $\mathfrak{t}$ is the smallest ordinal α such that not all towers of length α are extendible.

Definition (van Douwen [1984])

An *unbounded family* is a family F of functions $g\colon \omega \longrightarrow \omega$ such that no single function $h\colon \omega \longrightarrow \omega$ eventually dominates all members of F. The *unbounding number* (sometimes called the bounding number) $\mathfrak{b}$ is the smallest cardinality of an unbounded family.

Also recall that $\mathrm{cov}(\mathcal{M})$ denotes the minimal number of meagre sets of reals necessary to cover the reals.

Definition (van Douwen [1984])

A *splitting family* is a family F of sets of natural numbers such that for every infinite set x of natural numbers, there is a member of F splitting x. The *splitting number* $\mathfrak{s}$ is the smallest cardinality of a splitting family.

Definition

A *countably splitting family* is a family F of sets of natural numbers such that for every countable collection X of infinite sets of natural numbers, there is a member of F splitting every element of X. The *countably splitting number* $\mathfrak{s}_{\aleph_0}$ is the smallest cardinality of a countably splitting family.

Observation

$\mathfrak{s} \leqslant \mathfrak{s}_{\aleph_0}$.

Proposition ([Kamburelis and Węglorz, 1996, Proposition 2.1])

$\mathfrak{s}_{\aleph_0} \leqslant \max(\mathfrak{b}, \mathfrak{s})$.

Proposition ([Kamburelis and Węglorz, 1996, Proposition 2.3])

$\min(\mathrm{cov}(\mathcal{M}), \mathfrak{s}_{\aleph_0}) \leqslant \mathfrak{s}$.

Question

Is $\mathfrak{s} < \mathfrak{s}_{\aleph_0}$ *consistent?*

Definition ([Brendle and Raghavan, 2014, Definition 31])

A *tail-splitting sequence* is a sequence $\langle a_\alpha \mid \alpha < \kappa \rangle$ of sets of natural numbers such that for every infinite set x of natural numbers there is an $\alpha < \kappa$ such that a_β splits x for all $\beta \in \kappa \setminus \alpha$. The *tail splitting number* $\mathfrak{s}_{tail}$ is the shortest length of a tail-splitting sequence.

Theorem ([Brendle and Raghavan, 2014, Theorem 40])

$\mathfrak{s} < \mathfrak{s}_{tail}$ *is consistent.*

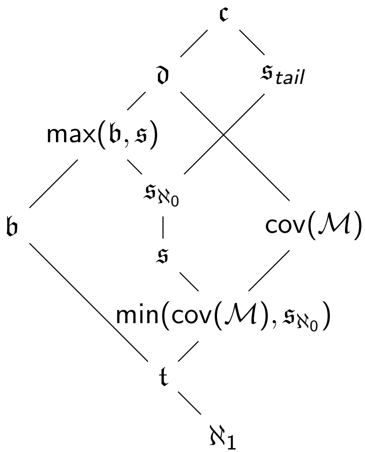

Figure: The inequalities between the aforementioned cardinal characteristics known to be ZFC-provable.

Theorem (Erdős and Rado [1956])

$$\begin{pmatrix}\omega_1\\ \omega\end{pmatrix} \longrightarrow \begin{pmatrix}\omega_1 & \omega\\ \omega & \omega\end{pmatrix}.$$

Theorem (Szemerédi, unpublished)

Martin's Axiom implies $\begin{pmatrix}\mathfrak{c}\\ \omega\end{pmatrix} \longrightarrow \begin{pmatrix}\mathfrak{c} & \kappa\\ \omega & \omega\end{pmatrix}$ *for all cardinals* $\kappa < \mathfrak{c}$.

Theorem (Jones [2008])

$$\begin{pmatrix} \kappa \\ \omega \end{pmatrix} \longrightarrow \begin{pmatrix} \kappa & \alpha \\ \omega & \omega \end{pmatrix} \text{ for any regular uncountable } \kappa \leqslant \mathfrak{c}$$

$$\text{and all } \alpha < \min(\mathfrak{p}, \kappa).$$

Theorem (Jones [2008])

$$\begin{pmatrix}\kappa\\ \omega\end{pmatrix} \longrightarrow \begin{pmatrix}\kappa & \alpha\\ \omega & \omega\end{pmatrix}$$ *for any regular uncountable* $\kappa \leqslant \mathfrak{c}$

and all $\alpha < \min(\mathfrak{p}, \kappa)$.

Theorem (Malliaris and Shelah [2013])

$\mathfrak{p} = \mathfrak{t}$.

Proposition ([Garti and Shelah, 2012, Claim 1.4])

If $\aleph_1 < \mathfrak{s}$, *then* $\begin{pmatrix}\omega_1\\ \omega\end{pmatrix} \longrightarrow \begin{pmatrix}\omega_1\\ \omega\end{pmatrix}_2$.

Question ([Garti and Shelah, 2014, Question 1.7(a)])

Is it consistent that $\mathfrak{p} = \mathfrak{s}$ *and* $\begin{pmatrix}\mathfrak{p}\\ \omega\end{pmatrix} \longrightarrow \begin{pmatrix}\mathfrak{p}\\ \omega\end{pmatrix}_2$?

Observation (Brendle and Raghavan [2014])

The following are equivalent:

$$\binom{\lambda}{\omega} \longrightarrow \binom{\lambda}{\omega}_2 \tag{1}$$

$$\mathrm{cf}(\lambda) \neq \omega \text{ and } \lambda < \mathfrak{s}_{tail}. \tag{2}$$

Corollary ([Brendle and Raghavan, 2014, Corollary 45])

It is consistent that $\mathfrak{s} = \aleph_1$ *while* $\binom{\omega_1}{\omega} \longrightarrow \binom{\omega_1}{\omega}_2$.

Theorem (Klausner and W.)

$$\binom{\kappa}{\eta} \longrightarrow \binom{\kappa\ \alpha}{\eta\ \eta} \text{ for any cardinal } \kappa \leqslant \mathfrak{c} \text{ of}$$

uncountable cofinality and all $\alpha < \min(\mathfrak{t}, \mathrm{cf}(\kappa))$.

Corollary

$$\binom{\kappa}{\omega} \longrightarrow \binom{\kappa\ \alpha}{\omega\ \omega} \text{ for any cardinal } \kappa \leqslant \mathfrak{c} \text{ of}$$

uncountable cofinality and all $\alpha < \min(\mathfrak{t}, \mathrm{cf}(\kappa))$.

Proposition (Klausner and W.)

If $\kappa < \mathfrak{b}$ is a cardinal of uncountable cofinality while n is a natural number and $\alpha \leqslant \kappa$, then

$$\begin{pmatrix}\kappa\\ \omega^n\end{pmatrix} \longrightarrow \begin{pmatrix}\kappa & \alpha\\ \omega^n & \omega^n\end{pmatrix} \text{ if and only if } \begin{pmatrix}\kappa\\ \omega\end{pmatrix} \longrightarrow \begin{pmatrix}\kappa & \alpha\\ \omega & \omega\end{pmatrix}.$$

Corollary (Klausner and W.)

If κ is a regular uncountable cardinal smaller than $\mathfrak{b}$ while $\beta \in \omega^\omega \setminus \omega$ is additively indecomposable and $\alpha < \min(\mathfrak{t}, \kappa)$, then

$$\begin{pmatrix}\kappa\\ \beta\end{pmatrix} \longrightarrow \begin{pmatrix}\kappa & \alpha\\ \beta & \beta\end{pmatrix}.$$

Proposition ([Orr, 1995, Proposition 2])

Let A be a countable linearly ordered set and for every $a \in A$ let L_a be a finite linearly ordered set. Then there is an increasing map

$$\sigma : A \longrightarrow L = \sum_{a \in A} L_a$$

which maps onto all but finitely many points of L, and, in any event, onto at least one point in every L_a.

Theorem (Klausner and W.)

If α is an ordinal of cofinality $\mathfrak{b}$ and φ is a countable typewise decomposable order type, then

$$\binom{\alpha}{\varphi} \not\to \binom{\alpha\ 1}{\varphi\ \varphi}.$$

Corollary

Let φ be a countable order type. If φ is equimorphic to an order type in $\{0, 1, \omega^*, \omega, \eta\}$, then

$$\binom{\mathfrak{b}}{\varphi} \longrightarrow \binom{\mathfrak{b}\ \alpha}{\varphi\ \varphi} \text{ for all } \alpha < \mathfrak{t};$$

$$\text{otherwise } \binom{\mathfrak{b}}{\varphi} \not\to \binom{\mathfrak{b}\ 1}{\varphi\ \varphi}.$$

Question

$$\text{Does the relation } \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \longrightarrow \begin{pmatrix} \varphi & n \\ \psi & n \end{pmatrix}$$

hold for all countable unionwise indecomposable order types φ, ψ *and all natural numbers* n*?*

Question

$$\text{Does the relation } \begin{pmatrix} \omega_1 \\ \varphi \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \alpha \\ \varphi & \varphi \end{pmatrix}$$

necessarily hold for all countable ordinals α *and all countable unionwise indecomposable order types* φ*?*

Question

Is it consistent that

$$\begin{pmatrix}\omega_1\\ \varphi\end{pmatrix} \longrightarrow \begin{pmatrix}\omega_1 & \alpha\\ \varphi & \varphi\end{pmatrix}$$

for all countable ordinals α *and all countable unionwise indecomposable order types* φ*?*

Question

Does the relation $\begin{pmatrix}\kappa \\ \omega\end{pmatrix} \longrightarrow \begin{pmatrix}\kappa & \alpha \\ \omega & \omega\end{pmatrix}$

hold for all cardinals $\kappa \leqslant \mathfrak{c}$ *of uncountable cofinality and all* $\alpha < \min(\mathfrak{s}_{\aleph_0}, \mathrm{cf}(\kappa))$*?*

Thank u_4 your attention!

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