Definability aspects of ultrafilters

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$\mathcal{U}$ is closed under finite modification $\Rightarrow \mathcal{U}$ has measure 0 or 1, meager or comeager. But $\mathcal{U}$ and $\{\omega \setminus U : U \in \mathcal{U}\}$ partition $\mathcal{P}(\omega) \Rightarrow$ contradiction.

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3. Analytic filters are bounded, i.e. contained in a σ-compact $\subseteq [\omega]^\omega$.
4. There is a Σ^1_2 ultrafilter in L.
5. In fact any Σ^1_n ultrafilter is already Δ^1_n ($U \in \mathcal{U}$ iff $\omega \setminus U \notin \mathcal{U}$).

Ultrafilter bases

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A. Miller streamlined a technique for constructing various combinatorial families of reals in a Π^1_1 way in L. For instance he constructed a Π^1_1 mad family, independent family, Hamel basis, ... in L.

Using this technique we could show the following:

Theorem

(V=L)

1. *There is a* Π^1_1 *P-point base.*
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In strong contrast we showed:

Theorem

There is no Π^1_1 base for a Ramsey ultrafilter.

Recall that $\mathcal{U}$ is Ramsey iff $\mathcal{U}$ is a P- and a Q-point.

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Main ingredients:

Lemma

Let $\mathcal{U}$ *be Ramsey and* $M \preccurlyeq H(\theta)$ *countable where* $\mathcal{U} \in M$. *Then there is* $x \in \mathcal{U}$ *so that every* $y \subseteq x$ *is generic over* M.

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Lemma

Whenever X is Π^1_1, $Y \subseteq X$ with $\sup\{\omega_1^y : y \in Y\} < \omega_1$, then there is Y' Δ^1_1 such that $Y \subseteq Y' \subseteq X$.

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Theorem (Shelah)

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Corollary

It is consistent that every P-point is Ramsey and Δ^1_2 (in particular there is no Π^1_1 base for a P-point).

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In fact, whenever $\mathcal{U}$ is Δ^1_2, then $\mathcal{U} \otimes \mathcal{U}$ has a Π^1_1 base.

$$\mathcal{U} \otimes \mathcal{U} = \{x \subseteq \omega \times \omega : \{n \in \omega : \{m \in \omega : (n, m) \in x\} \in \mathcal{U}\} \in \mathcal{U}\}.$$

The Borel ultrafilter number

$$\mathfrak{r} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is a ... family}\}$$

then

$$\mathfrak{r}_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \Delta^1_1, \bigcup \mathcal{B} \text{ is a ... family}\}$$

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$\aleph_1 \leq \mathfrak{u}_B \leq \mathfrak{u} \leq \mathfrak{c}$.

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Observation

$\aleph_1 \leq \mathfrak{u}_B \leq \mathfrak{u} \leq \mathfrak{c}$. In fact $\mathfrak{b} \leq \mathfrak{u}_B$.
$\exists \Delta^1_2$ ultrafilter $\Rightarrow \mathfrak{u}_B = \aleph_1$.

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Corollary

After adding ω_1 many Cohen/Random/Silver reals there is no projective ultrafilter.

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Definition

$\mathbb{Q}$ has the mutual genericity property iff for any $M \preccurlyeq H(\theta)$ countable, $p, \mathbb{Q} \in M$, there is a master condition $q \leq p$ so that for any filters $G_0, \ldots, G_k$ contaning q, generic over M and pairwise different over M,

$$G_0 \times \cdots \times G_k \text{ is } \mathbb{Q}^{k+1} \text{ generic over } M.$$

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(CH) There is a collection $\mathcal{A}$ of Borel sets so that for any Suslin forcing $\mathbb{Q}$ with the mgp, $V^{\mathbb{Q}} \models \bigcup \mathcal{A}$ is an ultrafilter.
(V=L) There is a (lightface) Π^1_1 ultrafilter base X so for any ... $V^{\mathbb{Q}} \models X$ is an ultrafilter base.

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Remark

The theorem also applies to maximal independent and maximal almost disjoint families.

References

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