Heavy liftings

Alessandro Vignati
KU Leuven, Belgium

Winter School in Abstract Analysis - Hejnice - early 2019

(Not so) heavy liftings

Alessandro Vignati
KU Leuven, Belgium

Winter School in Abstract Analysis - Hejnice - early 2019

- H is the separable complex Hilbert space

$$\ell_2(\mathbb{N}) = \{(a_n) \mid a_n \in \mathbb{C},\ \sum |a_n|^2 < \infty\}.$$

It is normed by $\|\bar{a}\|^2 = \sum |a_n|^2$.

- H is the separable complex Hilbert space

$$\ell_2(\mathbb{N}) = \{(a_n) \mid a_n \in \mathbb{C},\ \sum |a_n|^2 < \infty\}.$$

It is normed by $\|\bar{a}\|^2 = \sum |a_n|^2$.

- $\mathcal{B}(H)$ is the Banach algebra of linear bounded operators $H \to H$, normed by

$$\|T\| = \sup_{\xi \in H, \|\xi\|=1} \|T\xi\|.$$

Multiplication is just composition. $\mathcal{B}(H)$ has a * operation $T \to T^*$ which is an isometric involution and satisfies $\|T\|^2 = \|TT^*\|$.

- H is the separable complex Hilbert space

$$\ell_2(\mathbb{N}) = \{(a_n) \mid a_n \in \mathbb{C},\ \sum |a_n|^2 < \infty\}.$$

It is normed by $\|\bar{a}\|^2 = \sum |a_n|^2$.

- $\mathcal{B}(H)$ is the Banach algebra of linear bounded operators $H \to H$, normed by

$$\|T\| = \sup_{\xi \in H, \|\xi\|=1} \|T\xi\|.$$

Multiplication is just composition. $\mathcal{B}(H)$ has a * operation $T \to T^*$ which is an isometric involution and satisfies $\|T\|^2 = \|TT^*\|$. This makes it a C^*-algebra.

- H is the separable complex Hilbert space

$$\ell_2(\mathbb{N}) = \{(a_n) \mid a_n \in \mathbb{C},\ \sum |a_n|^2 < \infty\}.$$

  It is normed by $\|\bar{a}\|^2 = \sum |a_n|^2$.

- $\mathcal{B}(H)$ is the Banach algebra of linear bounded operators $H \to H$, normed by

$$\|T\| = \sup_{\xi \in H, \|\xi\|=1} \|T\xi\|.$$

  Multiplication is just composition. $\mathcal{B}(H)$ has a * operation $T \to T^*$ which is an isometric involution and satisfies $\|T\|^2 = \|TT^*\|$. This makes it a C^*-algebra.

- In general, a C^*-algebra is a *-closed Banach subalgebra of $\mathcal{B}(H)$, for some Hilbert space H.

- H is the separable complex Hilbert space

$$\ell_2(\mathbb{N}) = \{(a_n) \mid a_n \in \mathbb{C},\ \sum |a_n|^2 < \infty\}.$$

  It is normed by $\|\bar{a}\|^2 = \sum |a_n|^2$.

- $\mathcal{B}(H)$ is the Banach algebra of linear bounded operators $H \to H$, normed by

$$\|T\| = \sup_{\xi \in H, \|\xi\|=1} \|T\xi\|.$$

  Multiplication is just composition. $\mathcal{B}(H)$ has a * operation $T \to T^*$ which is an isometric involution and satisfies $\|T\|^2 = \|TT^*\|$. This makes it a C^*-algebra.

- In general, a C^*-algebra is a *-closed Banach subalgebra of $\mathcal{B}(H)$, for some Hilbert space H.

- Equivalently, a C^*-algebra is a Banach algebra A with an isometric involution * such that $\|a\|^2 = \|aa^*\|$ whenever $a \in A$. (i.e., Every C^*-algebra in this abstract sense can be isometrically represented as a *-closed Banach subalgebra of $\mathcal{B}(H)$, for some H).

Examples:

- $M_n(\mathbb{C}) = \mathcal{B}(\ell_2(n))$, for all n. $\mathcal{B}(H)$ for all Hilbert H.

Examples:

- $M_n(\mathbb{C}) = \mathcal{B}(\ell_2(n))$, for all n. $\mathcal{B}(H)$ for all Hilbert H.
- If X is a locally compact Hausdorff space,

$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is cont. and vanishes at } \infty\}$$

  is a C^*-algebra. (sup-norm, operations are performed pointwise, involution is pointwise complex conjugation). $C_0(X)$ unital if and only if X compact.

Examples:

- $M_n(\mathbb{C}) = \mathcal{B}(\ell_2(n))$, for all n. $\mathcal{B}(H)$ for all Hilbert H.
- If X is a locally compact Hausdorff space,

$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is cont. and vanishes at } \infty\}$$

  is a C^*-algebra. (sup-norm, operations are performed pointwise, involution is pointwise complex conjugation). $C_0(X)$ unital if and only if X compact.
- Let A_n be C^*-algebras. Then

$$\prod A_n = \{(a_n) \mid a_n \in A_n, \sup_n \|a_n\| < \infty\}$$

  and

$$\bigoplus A_n = \{(a_n) \mid a_n \in A_n, \|a_n\| \to 0\}$$

  are C^*-algebras. (sup norm, coordinatewise operations).

Examples:

- $M_n(\mathbb{C}) = \mathcal{B}(\ell_2(n))$, for all n. $\mathcal{B}(H)$ for all Hilbert H.
- If X is a locally compact Hausdorff space,

$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is cont. and vanishes at } \infty\}$$

  is a C^*-algebra. (sup-norm, operations are performed pointwise, involution is pointwise complex conjugation). $C_0(X)$ unital if and only if X compact.
- Let A_n be C^*-algebras. Then

$$\prod A_n = \{(a_n) \mid a_n \in A_n, \sup_n \|a_n\| < \infty\}$$

  and

$$\bigoplus A_n = \{(a_n) \mid a_n \in A_n, \|a_n\| \to 0\}$$

  are C^*-algebras. (sup norm, coordinatewise operations).
- If $A \subseteq B$ are C^*-algebras, and $BAB \subseteq A$, A is an ideal. The quotient B/A is a C^*-algebra.

Examples:

- $M_n(\mathbb{C}) = \mathcal{B}(\ell_2(n))$, for all n. $\mathcal{B}(H)$ for all Hilbert H.
- If X is a locally compact Hausdorff space,

$$C_0(X) = \{f\colon X \to \mathbb{C} \mid f \text{ is cont. and vanishes at } \infty\}$$

  is a C^*-algebra. (sup-norm, operations are performed pointwise, involution is pointwise complex conjugation). $C_0(X)$ unital if and only if X compact.
- Let A_n be C^*-algebras. Then

$$\prod A_n = \{(a_n) \mid a_n \in A_n, \sup_n \|a_n\| < \infty\}$$

  and

$$\bigoplus A_n = \{(a_n) \mid a_n \in A_n, \|a_n\| \to 0\}$$

  are C^*-algebras. (sup norm, coordinatewise operations).
- If $A \subseteq B$ are C^*-algebras, and $BAB \subseteq A$, A is an ideal. The quotient B/A is a C^*-algebra. If A_n are C^*-algebras,

$$\prod A_n / \bigoplus A_n$$

  is a C^*-algebra, called the **reduced product** of the A_n's.

- Take $H = \ell_2(\mathbb{N})$. Let

$$\mathcal{F}(H) = \{T\colon H \to H \mid T[H] \text{ is finite dimensional}\}.$$

- Take $H = \ell_2(\mathbb{N})$. Let

$$\mathcal{F}(H) = \{T \colon H \to H \mid T[H] \text{ is finite dimensional}\}.$$

This is a subalgebra, but it is not $\|\cdot\|$-closed!

- Take $H = \ell_2(\mathbb{N})$. Let

$$\mathcal{F}(H) = \{T \colon H \to H \mid T[H] \text{ is finite dimensional}\}.$$

This is a subalgebra, but it is not $\|\cdot\|$-closed! Close it.

$$\mathcal{K}(H) = \overline{\mathcal{F}(H)}.$$

This is the algebra of compact operators.

- Take $H = \ell_2(\mathbb{N})$. Let

$$\mathcal{F}(H) = \{T \colon H \to H \mid T[H] \text{ is finite dimensional}\}.$$

This is a subalgebra, but it is not $\|\cdot\|$-closed! Close it.

$$\mathcal{K}(H) = \overline{\mathcal{F}(H)}.$$

This is the algebra of compact operators.

- $\mathcal{K}(H) \subseteq \mathcal{B}(H)$ is an ideal. The quotient

$$\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$$

is called the **Calkin algebra**. $\pi \colon \mathcal{B}(H) \to \mathcal{Q}(H)$ is the quotient map.

- Take $H = \ell_2(\mathbb{N})$. Let

$$\mathcal{F}(H) = \{T\colon H \to H \mid T[H] \text{ is finite dimensional}\}.$$

  This is a subalgebra, but it is not $\|\cdot\|$-closed! Close it.

$$\mathcal{K}(H) = \overline{\mathcal{F}(H)}.$$

  This is the algebra of compact operators.
- $\mathcal{K}(H) \subseteq \mathcal{B}(H)$ is an ideal. The quotient

$$\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$$

  is called the **Calkin algebra**. $\pi\colon \mathcal{B}(H) \to \mathcal{Q}(H)$ is the quotient map.

$\mathcal{Q}(H)$ is the noncommutative analog of $\ell_\infty/c_0 = C(\beta\omega \setminus \omega)$, and therefore its poset of projections is the noncommutative analog of $\mathcal{P}(\omega)/\operatorname{Fin}$.

Question

Which C^**-algebras embed into* $\mathcal{Q}(H)$*? Which reduced products?*

Question

Which C^*-algebras embed into $\mathcal{Q}(H)$? Which reduced products?

Fix a sequence $(n_k)_k \subseteq \omega$. Fix $\{I_k\}$ a partition on ω such that $|I_k| = n_k$, and e_i a base for $\ell_2(\mathbb{N})$. Then $\mathcal{B}(\text{span}\{e_i \mid i \in I_k\}) \cong M_{n_k}$. This defines an embedding $\prod M_{n_k} \to \mathcal{B}(H)$.

Question

Which C^*-algebras embed into $\mathcal{Q}(H)$? Which reduced products?

Fix a sequence $(n_k)_k \subseteq \omega$. Fix $\{I_k\}$ a partition on ω such that $|I_k| = n_k$, and e_i a base for $\ell_2(\mathbb{N})$. Then $\mathcal{B}(\text{span}\{e_i \mid i \in I_k\}) \cong M_{n_k}$. This defines an embedding $\prod M_{n_k} \to \mathcal{B}(H)$. Note that

$$\prod M_{n_k} \cap \mathcal{K}(H) = \bigoplus M_{n_k},$$

so this defines a unital embedding

$$\prod M_{n_k} / \bigoplus M_{n_k} \to \mathcal{Q}(H).$$

Question

Which C^*-algebras embed into $\mathcal{Q}(H)$? Which reduced products?

Fix a sequence $(n_k)_k \subseteq \omega$. Fix $\{I_k\}$ a partition on ω such that $|I_k| = n_k$, and e_i a base for $\ell_2(\mathbb{N})$. Then $\mathcal{B}(\mathrm{span}\{e_i \mid i \in I_k\}) \cong M_{n_k}$. This defines an embedding $\prod M_{n_k} \to \mathcal{B}(H)$. Note that

$$\prod M_{n_k} \cap \mathcal{K}(H) = \bigoplus M_{n_k},$$

so this defines a unital embedding

$$\prod M_{n_k} / \bigoplus M_{n_k} \to \mathcal{Q}(H).$$

In case one considers infinite-dimensional (as vector spaces) C^*-algebras A_n, there is no way of obtaining such a *nice* embedding

$$\phi\colon \prod A_n / \bigoplus A_n \to \mathcal{Q}(H).$$

(Here nice means: there are mutually orthogonal $\phi_n\colon A_n \to \mathcal{B}(H)$ such that $\phi = \pi(\sum \phi_n)$).

Question

Which C^*-algebras embed into $\mathcal{Q}(H)$?

Question

Which C^*-algebras embed into $\mathcal{Q}(H)$?

There is no easy way to embed many reduced products into $\mathcal{Q}(H)$. On the other hand....

Question

Which C^*-algebras embed into $\mathcal{Q}(H)$?

There is no easy way to embed many reduced products into $\mathcal{Q}(H)$. On the other hand....

Theorem (Farah-Hirshberg-V.)

All C^*-algebras of density $\aleph_1$ embed into $\mathcal{Q}(H)$.

Question

Which C^*-algebras embed into $\mathcal{Q}(H)$?

There is no easy way to embed many reduced products into $\mathcal{Q}(H)$. On the other hand....

Theorem (Farah-Hirshberg-V.)

All C^*-algebras of density $\aleph_1$ embed into $\mathcal{Q}(H)$. So, under CH, *if A_n is separable for all n, there is an embedding*

$$\prod A_n / \bigoplus A_n \to \mathcal{Q}(H).$$

Moreover, if X is a locally compact second countable space, then $C(\beta X \setminus X)$ embeds into the Calkin algebra.

We turn to nonembedding.

We turn to nonembedding.

Definition

The Proper Forcing Axiom PFA is a Forcing Axiom introduced by Shelah as a generalization of the Baire Category Theorem. It asserts that a large class of forcings (proper ones) admits generic filters intersecting $\aleph_1$-many dense sets. It extends Martin's Axiom at level $\aleph_1$, implies $\mathfrak{c} = \omega_2$, and implies Todorcevic's Open Colouring Axiom.

We turn to nonembedding.

Definition

The Proper Forcing Axiom PFA is a Forcing Axiom introduced by Shelah as a generalization of the Baire Category Theorem. It asserts that a large class of forcings (proper ones) admits generic filters intersecting $\aleph_1$-many dense sets. It extends Martin's Axiom at level $\aleph_1$, implies $\mathfrak{c} = \omega_2$, and implies Todorcevic's Open Colouring Axiom.

Why do we believe that PFA implies that there are no embeddings of certain C^*-algebras into $\mathcal{Q}(H)$?

We turn to nonembedding.

Definition

The Proper Forcing Axiom PFA is a Forcing Axiom introduced by Shelah as a generalization of the Baire Category Theorem. It asserts that a large class of forcings (proper ones) admits generic filters intersecting $\aleph_1$-many dense sets. It extends Martin's Axiom at level $\aleph_1$, implies $\mathfrak{c} = \omega_2$, and implies Todorcevic's Open Colouring Axiom.

Why do we believe that PFA implies that there are no embeddings of certain C^*-algebras into $\mathcal{Q}(H)$? Because we said that $\mathcal{Q}(H)$ and $C(\beta\omega \setminus \omega)$ look very much alike! and...

We turn to nonembedding.

Definition

The Proper Forcing Axiom PFA is a Forcing Axiom introduced by Shelah as a generalization of the Baire Category Theorem. It asserts that a large class of forcings (proper ones) admits generic filters intersecting $\aleph_1$-many dense sets. It extends Martin's Axiom at level $\aleph_1$, implies $\mathfrak{c} = \omega_2$, and implies Todorcevic's Open Colouring Axiom.

Why do we believe that PFA implies that there are no embeddings of certain C^*-algebras into $\mathcal{Q}(H)$? Because we said that $\mathcal{Q}(H)$ and $C(\beta\omega \setminus \omega)$ look very much alike! and...

Definition

A topological space X is said not to look like ω if it is locally compact noncompact and second countable, and there is no compact $K \subseteq X$ for which $X \setminus K \cong \omega$.

We turn to nonembedding.

Definition

The Proper Forcing Axiom PFA is a Forcing Axiom introduced by Shelah as a generalization of the Baire Category Theorem. It asserts that a large class of forcings (proper ones) admits generic filters intersecting $\aleph_1$-many dense sets. It extends Martin's Axiom at level $\aleph_1$, implies $\mathfrak{c} = \omega_2$, and implies Todorcevic's Open Colouring Axiom.

Why do we believe that PFA implies that there are no embeddings of certain C^*-algebras into $\mathcal{Q}(H)$? Because we said that $\mathcal{Q}(H)$ and $C(\beta\omega \setminus \omega)$ look very much alike! and...

Definition

A topological space X is said not to look like ω if it is locally compact noncompact and second countable, and there is no compact $K \subseteq X$ for which $X \setminus K \cong \omega$.

Theorem (Dow-Hart)

Assume PFA*. Suppose that X does not look like ω. Then $C(\beta X \setminus X)$ does not embed into $C(\beta\omega \setminus \omega)$. (but they do under* CH*!)*

How do we generalize this result? Via **heavy liftings**!

How do we generalize this result? Via **heavy liftings**! Suppose that $A \subseteq B$ is an ideal with quotient map $\pi_A \colon B \to B/A$. Let

$$\phi \colon B/A \to \mathcal{Q}(H)$$

be an embedding.

How do we generalize this result? Via **heavy liftings**! Suppose that $A \subseteq B$ is an ideal with quotient map $\pi_A \colon B \to B/A$. Let

$$\phi \colon B/A \to \mathcal{Q}(H)$$

be an embedding.

$$\begin{array}{ccc} B & & \mathcal{B}(H) \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi} \\ B/A & \xrightarrow[\phi]{} & \mathcal{Q}(H) \end{array}$$

The idea is to use PFA to show that **whenever** $\phi\colon B/A \to \mathcal{Q}(H)$ is an embedding, it is possible to find a **nice** lifting $\Phi\colon B \to \mathcal{B}(H)$

$$\begin{array}{ccc} B & \xrightarrow{\ \Phi\ } & \mathcal{B}(H) \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi} \\ B/A & \xrightarrow[\ \phi\ ]{} & \mathcal{Q}(H) \end{array}$$

The idea is to use PFA to show that **whenever** $\phi\colon B/A \to \mathcal{Q}(H)$ is an embedding, it is possible to find a **nice** lifting $\Phi\colon B \to \mathcal{B}(H)$

$$\begin{array}{ccc} B & \xrightarrow{\ \Phi\ } & \mathcal{B}(H) \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi} \\ B/A & \xrightarrow[\ \phi\ ]{} & \mathcal{Q}(H) \end{array}$$

Then, studying B and A, we want to use ZFC to show that nice maps $\Phi\colon B \to A$ such that $\Phi(B) \cap \mathcal{K}(H) = A$ cannot exist.

Proposition

- *Let A_n be infinite-dimensional C^*-algebras. Then there is no sequence of mutually orthogonal maps $\phi_n \colon A \to \mathcal{B}(H)$ such that $\Phi = \sum \phi_n$ is a lifting for an embedding $\prod A_n / \bigoplus A_n \to \mathcal{Q}(H)$.*

Proposition

- *Let* A_n *be infinite-dimensional* C^**-algebras. Then there is no sequence of mutually orthogonal maps* $\phi_n \colon A \to \mathcal{B}(H)$ *such that* $\Phi = \sum \phi_n$ *is a lifting for an embedding* $\prod A_n / \bigoplus A_n \to \mathcal{Q}(H)$.
- *Suppose that* X *does not look like* ω*. There is a notion of niceness such that there is no nice map* $\Phi \colon C(\beta X) \to \mathcal{B}(H)$ *can be a lifting for an embedding* $C(\beta X \setminus X) \cong C(\beta X)/C_0(X) \to \mathcal{Q}(H)$.

Proposition

- *Let A_n be infinite-dimensional C^*-algebras. Then there is no sequence of mutually orthogonal maps $\phi_n \colon A \to \mathcal{B}(H)$ such that $\Phi = \sum \phi_n$ is a lifting for an embedding $\prod A_n / \bigoplus A_n \to \mathcal{Q}(H)$.*
- *Suppose that X does not look like ω. There is a notion of niceness such that there is no nice map $\Phi \colon C(\beta X) \to \mathcal{B}(H)$ can be a lifting for an embedding $C(\beta X \setminus X) \cong C(\beta X)/C_0(X) \to \mathcal{Q}(H)$.*

So we just need to lift embeddings to nice maps, and we're done!

Theorem (McKenney-V.)

Assume PFA. *Let* $\phi\colon \ell_\infty/c_0 \to \mathcal{Q}(H)$ *be an injection. Then there are* **-homomorphisms* $\phi_n\colon \mathbb{C} \to \mathcal{B}(H)$ *and a nonmeager dense ideal* $\mathscr{I} \subseteq \mathcal{P}(\omega)$ *such that* $\Phi = \sum \phi_n\colon \ell_\infty \to \mathcal{B}(H)$ *lifts* ϕ *on elements whose support is in* $\mathscr{I}$.

Theorem (McKenney-V.)

Assume PFA. *Let* $\phi\colon \ell_\infty/c_0 \to \mathcal{Q}(H)$ *be an injection. Then there are* **-homomorphisms* $\phi_n\colon \mathbb{C} \to \mathcal{B}(H)$ *and a nonmeager dense ideal* $\mathscr{I} \subseteq \mathcal{P}(\omega)$ *such that* $\Phi = \sum \phi_n\colon \ell_\infty \to \mathcal{B}(H)$ *lifts* ϕ *on elements whose support is in* $\mathscr{I}$*.*

Note that $\ell_\infty/c_0 \cong \prod \mathbb{C}/\bigoplus \mathbb{C}$. We want to replace $\mathbb{C}$ with a finite-dimensional object, and we do so.

Theorem (McKenney-V.)

Assume PFA. *Let* $\phi\colon \ell_\infty/c_0 \to \mathcal{Q}(H)$ *be an injection. Then there are* **-homomorphisms* $\phi_n\colon \mathbb{C} \to \mathcal{B}(H)$ *and a nonmeager dense ideal* $\mathscr{I} \subseteq \mathcal{P}(\omega)$ *such that* $\Phi = \sum \phi_n\colon \ell_\infty \to \mathcal{B}(H)$ *lifts* ϕ *on elements whose support is in* $\mathscr{I}$.

Note that $\ell_\infty/c_0 \cong \prod \mathbb{C}/\bigoplus \mathbb{C}$. We want to replace $\mathbb{C}$ with a finite-dimensional object, and we do so.

Theorem (McKenney-V.)

Assume PFA. *Let* A_n *be unital* C^**-algebras,* $F_n \subseteq A_n$ *be finite-dimensional Banach spaces, and* $\phi\colon \prod A_n/\bigoplus A_n \to \mathcal{Q}(H)$ *be an embedding.*

Theorem (McKenney-V.)

Assume PFA. *Let* $\phi\colon \ell_\infty/c_0 \to \mathcal{Q}(H)$ *be an injection. Then there are* **-homomorphisms* $\phi_n\colon \mathbb{C} \to \mathcal{B}(H)$ *and a nonmeager dense ideal* $\mathscr{I} \subseteq \mathcal{P}(\omega)$ *such that* $\Phi = \sum \phi_n\colon \ell_\infty \to \mathcal{B}(H)$ *lifts* ϕ *on elements whose support is in* $\mathscr{I}$.

Note that $\ell_\infty/c_0 \cong \prod \mathbb{C}/\bigoplus \mathbb{C}$. We want to replace $\mathbb{C}$ with a finite-dimensional object, and we do so.

Theorem (McKenney-V.)

Assume PFA. *Let* A_n *be unital* C^**-algebras,* $F_n \subseteq A_n$ *be finite-dimensional Banach spaces, and* $\phi\colon \prod A_n/\bigoplus A_n \to \mathcal{Q}(H)$ *be an embedding. Then there are orthogonal maps* $\phi_n\colon F_n \to \mathcal{B}(H)$ *and a nonmeager dense ideal* $\mathscr{I}$ *such that* $\Phi = \sum \phi_n$ *lifts* ϕ *on elements in* $\prod F_n$ *whose support is in* $\mathscr{I}$.

Theorem (McKenney-V.+V.)

Assume PFA, *and let* A_n *be unital separable infinite-dimensional* C^**-algebras. Then* $\prod A_n / \bigoplus A_n$ *does not embed into* $\mathcal{Q}(H)$.

Theorem (McKenney-V.+V.)

Assume PFA, *and let A_n be unital separable infinite-dimensional* C^**-algebras. Then $\prod A_n / \bigoplus A_n$ does not embed into $\mathcal{Q}(H)$.*

Theorem (V.)

Assume PFA *and suppose that X does not look like ω. Then $C(\beta X \setminus X)$ does not embed into $\mathcal{Q}(H)$.*

Theorem (McKenney-V.+V.)

Assume PFA, *and let A_n be unital separable infinite-dimensional* C^**-algebras. Then $\prod A_n / \bigoplus A_n$ does not embed into $\mathcal{Q}(H)$.*

Theorem (V.)

Assume PFA *and suppose that X does not look like ω. Then $C(\beta X \setminus X)$ does not embed into $\mathcal{Q}(H)$.*

In fact a similar result can be proved for corona C^*-algebras, which are the noncommutative equivalents of Čech-Stone remainders. (i.e., the corona algebra of A embeds into $\mathcal{Q}(H)$ only whenever A looks like a subalgebra of $\mathcal{K}(H)$)

More things to say/to do:

- We do not need PFA, but usually a stronger version of OCA and Martin's Axiom at $\aleph_1$.

More things to say/to do:

- We do not need PFA, but usually a stronger version of OCA and Martin's Axiom at $\aleph_1$.
- These heavy lifting results are also used to treat automorphisms of corona algebras, and consequently homeomorphisms of spaces of the form $\beta X \setminus X$. They are also used to study mutual embeddings of algebras of the form $C(\beta X \setminus X)$ and $C(\beta Y \setminus Y)$, and of corona algebras.

More things to say/to do:

- We do not need PFA, but usually a stronger version of OCA and Martin's Axiom at $\aleph_1$.
- These heavy lifting results are also used to treat automorphisms of corona algebras, and consequently homeomorphisms of spaces of the form $\beta X \setminus X$. They are also used to study mutual embeddings of algebras of the form $C(\beta X \setminus X)$ and $C(\beta Y \setminus Y)$, and of corona algebras.
- Inspired by the equivalent result of Boolean algebras and $\mathcal{P}(\omega)/\operatorname{Fin}$, Farah, Katsimpas and Vaccaro recently proved that for a given C^*-algebra A one can find a ccc forcing $\mathbb{P}_A$ that forces A to embed in $\mathcal{Q}(H)$.

More things to say/to do:

- We do not need PFA, but usually a stronger version of OCA and Martin's Axiom at $\aleph_1$.
- These heavy lifting results are also used to treat automorphisms of corona algebras, and consequently homeomorphisms of spaces of the form $\beta X \setminus X$. They are also used to study mutual embeddings of algebras of the form $C(\beta X \setminus X)$ and $C(\beta Y \setminus Y)$, and of corona algebras.
- Inspired by the equivalent result of Boolean algebras and $\mathcal{P}(\omega)/\operatorname{Fin}$, Farah, Katsimpas and Vaccaro recently proved that for a given C^*-algebra A one can find a ccc forcing $\mathbb{P}_A$ that forces A to embed in $\mathcal{Q}(H)$.
- But such a poset is messy. Making it nicer is very much needed.

More things to say/to do:

- We do not need PFA, but usually a stronger version of OCA and Martin's Axiom at $\aleph_1$.
- These heavy lifting results are also used to treat automorphisms of corona algebras, and consequently homeomorphisms of spaces of the form $\beta X \setminus X$. They are also used to study mutual embeddings of algebras of the form $C(\beta X \setminus X)$ and $C(\beta Y \setminus Y)$, and of corona algebras.
- Inspired by the equivalent result of Boolean algebras and $\mathcal{P}(\omega)/\operatorname{Fin}$, Farah, Katsimpas and Vaccaro recently proved that for a given C^*-algebra A one can find a ccc forcing $\mathbb{P}_A$ that forces A to embed in $\mathcal{Q}(H)$.
- But such a poset is messy. Making it nicer is very much needed.
- Again comparing $\mathcal{Q}(H)$ to $\mathcal{P}(\omega)/\operatorname{Fin}$, it is not known if we can get that $\mathfrak{c}$ is large and all algebras of density $\mathfrak{c}$ embed into $\mathcal{Q}(H)$.

More things to say/to do:

- We do not need PFA, but usually a stronger version of OCA and Martin's Axiom at $\aleph_1$.
- These heavy lifting results are also used to treat automorphisms of corona algebras, and consequently homeomorphisms of spaces of the form $\beta X \setminus X$. They are also used to study mutual embeddings of algebras of the form $C(\beta X \setminus X)$ and $C(\beta Y \setminus Y)$, and of corona algebras.
- Inspired by the equivalent result of Boolean algebras and $\mathcal{P}(\omega)/\operatorname{Fin}$, Farah, Katsimpas and Vaccaro recently proved that for a given C^*-algebra A one can find a ccc forcing $\mathbb{P}_A$ that forces A to embed in $\mathcal{Q}(H)$.
- But such a poset is messy. Making it nicer is very much needed.
- Again comparing $\mathcal{Q}(H)$ to $\mathcal{P}(\omega)/\operatorname{Fin}$, it is not known if we can get that $\mathfrak{c}$ is large and all algebras of density $\mathfrak{c}$ embed into $\mathcal{Q}(H)$.
- We can also ask whether ultrapowers embed into $\mathcal{Q}(H)$. With arguments 'à la Woodin', this can lead to automatic continuity results.

More things to say/to do:

- We do not need PFA, but usually a stronger version of OCA and Martin's Axiom at $\aleph_1$.
- These heavy lifting results are also used to treat automorphisms of corona algebras, and consequently homeomorphisms of spaces of the form $\beta X \setminus X$. They are also used to study mutual embeddings of algebras of the form $C(\beta X \setminus X)$ and $C(\beta Y \setminus Y)$, and of corona algebras.
- Inspired by the equivalent result of Boolean algebras and $\mathcal{P}(\omega)/\mathrm{Fin}$, Farah, Katsimpas and Vaccaro recently proved that for a given C^*-algebra A one can find a ccc forcing $\mathbb{P}_A$ that forces A to embed in $\mathcal{Q}(H)$.
- But such a poset is messy. Making it nicer is very much needed.
- Again comparing $\mathcal{Q}(H)$ to $\mathcal{P}(\omega)/\mathrm{Fin}$, it is not known if we can get that $\mathfrak{c}$ is large and all algebras of density $\mathfrak{c}$ embed into $\mathcal{Q}(H)$.
- We can also ask whether ultrapowers embed into $\mathcal{Q}(H)$. With arguments 'à la Woodin', this can lead to automatic continuity results. Big problem here: the notion of gap in $\mathcal{Q}(H)$ is very complicated. There is an analytic gap in $\mathcal{Q}(H)$ which cannot be frozen by any ccc forcing!

Thanks!