Singular Compact Logics

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(Shelah) If every subset of size $< \kappa$ of a graph G of size κ has the coloring number $\leq \lambda < \kappa$, then so does G.

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If a Banach space has a biorthogonal sequence of ever length $< \kappa$, then it has a biorthogonal sequence of length κ.

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Some large cardinals

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A typical chain model A with decomposition $\langle A_n : n < \omega \rangle$ is denoted by $(A_n)_n$. It is mostly interesting when κ is a strong limit and $2^{|A_n|} < |A_{n+1}|$.

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The spirit here is that $L^c_{\kappa,\kappa}$ behaves very much like $L_{\omega_1,\omega}$.

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The orbit of a chain model A is always a Σ^1_1 set. The orbit is Δ^1_1 if and only if there is a tree T of height and size κ with no branches of length κ such that for any chain model B, player I has a winning strategy in $EFD^{c,<\kappa}_T(A, B)$ if and only if $A \approx^c B$.

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This theorem completed the classical analysis of the chain logic and found applications in generalised descriptive set theory through the work of Moto Ross et al.

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Proof by constructing a counterexample using the notion of a well order.

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We consider Chu spaces $(L, \models, S)$ where L is a set of sentences closed under conjunctions, S a set or a class of structures of the same signature as the sentences in L and $\models$ a relation between the elements of S and the elements of L, whose interpretation is a satisfaction relation which satisfies Tarski's definition of truth for the quantifier-free formulas.

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Theorem Suppose that $(L, \models, S) \leq (L', \models', S')$ and $(L', \models', S')$ is compact. Then so is $(L, \models, S)$.

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The lemma implies the density of g, since any model of $L_{\kappa,\omega}$ size κ can be represented as a proper chain model and the two will satisfy the same sentences. □

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Mirna Džamonja (in joint work with Jouko Väänänen, with Will Boney and Stamatis Dimopoulos)

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Theorem [Keisler 1968, Fuhrken 1965 for $\mathrm{cf}(\kappa) > \omega$] If κ is a strong limit singular, then both $\mathcal{L}^{\lessdot,\kappa}$ and $L(Q_\kappa)$ are compact.

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