On countably saturated linear orders

Ziemowit Kostana

University of Warsaw, Poland
and
Czech Academy of Sciences, Czech Republic

Winter School in Abstract Analysis, Hejnice, 29.01.2019
A linear order is compact, if it’s compact in the order topology. This means, it is Dedekind complete, and has both endpoints.

A linear order is linearly ordered continuum, if it is compact and connected in the order topology. This means, it is compact and dense.

$I = [-1, 1]$. 

Z. Kostana On countably saturated linear orders
Definition

We’ll say that a linear order \((L, \leq)\) is countably saturated, if for any countable linear orders \(a, b\), and increasing functions \(i : a \rightarrow b\), \(f : a \rightarrow L\), there exists \(\tilde{f} : b \rightarrow L\), such that \(\tilde{f} \circ i = f\).
**Definition**

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There exists an equivalent definition.

**Lemma**

Linear order is countably saturated if and only if

- it is dense, without endpoints,
- no countable increasing sequence has supremum,
- no countable decreasing sequence has infimum,
- there are no \((\omega, \omega)\)-gaps: for any two sequences \(\{x_n\}_{n<\omega}, \{y_n\}_{n<\omega}\) such that \(\forall n < \omega \ x_n < x_{n+1} < y_{n+1} < y_n\), there exists \(z\) s.t. \(\forall n < \omega \ x_n < z < y_n\).
Proposition

Any countably saturated linear order contains an isomorphic copy of the real line.

Proof.

Let \((L, \leq)\) be a countably saturated linear order. It is dense, so there exists an injection \(i : \mathbb{Q} \hookrightarrow L\). For any real number \(r\), we want to define \(i(r)\).

Notice that sets \(i[\{q \in \mathbb{Q} : q > r\}] > i[\{q \in \mathbb{Q} : q < r\}]\) are countable. Therefore, there exists \(l \in L\) such that

\[
i[\{q \in \mathbb{Q} : q > r\}] > l > i[\{q \in \mathbb{Q} : q < r\}].
\]

We define \(i(r) = l\).
Theorem (Hausdorff)

Assume \((L, \leq_L)\) is countably saturated, and \((X, \leq_X)\) doesn’t contain a copy of \(\omega_1\) or \(\omega_1^*\). Then exists an embedding \(i : X \hookrightarrow L\).
A countably saturated linear order $L$ is prime, if it embeds into any other countably saturated linear order.
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Example (Sierpiński)

Let $Q = \{x \in \{0, 1\}^{\omega_1} | \exists_{\alpha < \omega_1} x(\alpha) = 1, \forall \beta > \alpha x(\beta) = 0\}$, with lexicographic order. This order is prime countably saturated.
Definition

\[ \mathbb{L}^{\omega_1} = \{ x \in \mathcal{I}^{\omega_1} \mid \{ \alpha < \omega_1 : x(\alpha) \neq 0 \} \leq \omega \}, \]

with lexicographic order. If \( D \) is compact linear order, and \( d_0 \in D \) is neither least, nor greatest element of \( D \), then we define

\[ \mathbb{L}^{\omega_1}_{(D,d_0)} = \{ x \in D^{\omega_1} \mid \{ \alpha < \omega_1 : x(\alpha) \neq d_0 \} \leq \omega \}. \]
Theorem 1

\( \mathbb{L}^{\omega_1} \) and \( \mathbb{L}^{\omega_1}_{(D,d_0)} \) are countably saturated.

Theorem 2

\( \mathbb{L}^{\omega_1} \) is prime countably saturated. Moreover, if \( D \) is separable, compact, and \( d_0 \in D \) is neither the least, nor the greatest element, \( \mathbb{L}^{\omega_1}_{(D,d_0)} \) is prime.
Theorem (folklore)

Under CH, all countably saturated linear orders of cardinality $\mathfrak{c}$ are isomorphic.
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Under CH, all countably saturated linear orders of cardinality $c$ are isomorphic.

In fact, the category of countable linear orders with embeddings, has unique $\omega_1$-Fraïssé limit.
Theorem (foklore)

Without CH, no.
Proof.

\[ \mathbb{L}^{\omega_1} = \{ x \in I^{\omega_1} \mid \left| \{ \alpha < \omega_1 : x(\alpha) \neq 0 \} \right| \leq \omega \}, \]

and

\[ \{ x \in I^{\omega_2} \mid \left| \{ \alpha < \omega_2 : x(\alpha) \neq 0 \} \right| \leq \omega \}, \]

are both countably saturated. But the second contains a copy of \( \omega_2 \), while the first doesn’t.
But what if we want same better examples?
Example

*In the Cohen model there exists two non-isomorphic countably saturated linear orders of cardinality $\mathfrak{c}$, none of which contains copy of $\omega_2$ or $\omega_2^*$.***
Outline of the proof:
Let $M$ be a model of $CH$, $M[G]$ be extension by $Fn_{<\omega}(\omega_2)$.

- First example will be $\mathbb{L}^{\omega_1}$ (in $M[G]$). We show, that it doesn’t contain copy of any linear order of cardinality $\omega_2$, which was in $M$.

- For second example, we take $2^{\omega_1}$, and inductively define an increasing sequence of linear orders $\{R_\alpha\}_{\alpha \leq \omega_1}$, such that $R_0 = 2^{\omega_1}$, and $R_{\omega_1}$ is countably saturated.

\[
(2^{\omega_1})^M \subset 2^{\omega_1} \subseteq R_{\omega_1},
\]

so these two cannot be isomorphic.
We’ll use notion of dimension for better classification of linear orders.

**Definition (V. Novák, 1963)**

*Let $L$ and $X$ be linear orders. We define dimension of $X$ with respect to $L$ as:*

\[
\text{L-dim } X = \min\{\alpha \in ON | X \hookrightarrow L^\alpha\}.
\]
Let us write down some easy observations.

**Proposition**

*For any linear orders $L, L_1, L_2, X$, the following holds.*

- If $X_1 \hookrightarrow X_2$, then $L$-dim $X_1 \leq L$-dim $X_2$.
- If $L_1 \hookrightarrow L_2$, then $L_1$-dim $X \geq L_2$-dim $X$.
- If $L_1 \hookrightarrow L_2$ and $L_2 \hookrightarrow L_1$, then for every $X$, $L_1$-dim $X = L_2$-dim $X$. 
In particular, notions of $2^\omega$-dim $X$, I-dim $X$, and $\mathbb{R}$-dim $X$ coincide. We will denote them I-dim $X$.

**Theorem (Novotný, 1953; Novák, 1963)**

Let $L$ be a linearly ordered continuum. Then for any ordinal $\alpha$, $L^\alpha$ is a linearly ordered continuum, and $L$-dim $L^\alpha = \alpha$.

**Corollary**

If $\alpha$ is an ordinal with the property, that $\omega \cdot \alpha = \alpha$, then I-dim $2^\alpha = \alpha$. 
Assume $c = 2^{\omega_1}$. Let $X = I^{\omega_1}$. Then $\mathbb{L}^{\omega_1}_{(X,0)}$ is a countably saturated linear order of cardinality $c$, without copy of $\omega_2$ or $\omega_2^*$, and $I$-dim equal to $\omega^2_1$. In particular $\mathbb{L}^{\omega_1}_{(X,0)}$ is not isomorphic to $\mathbb{L}^{\omega_1}$. 
Proposition (Fleischer, 1961)

If $\text{I-dim } L < \omega_1$, then $L$ doesn’t contain a copy of $\omega_1$ or $\omega_1^*$. 
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*If* $\text{I-dim } L < \omega_1$, *then* $L$ *doesn’t contain a copy of* $\omega_1$ *or* $\omega^*_1$.

The converse doesn’t hold, though the false proof was published in

Proposition

Let \((L, \leq)\) be countably saturated linear order. The following are equivalent:

- \(L\) is prime.
- \(L = \bigcup_{\alpha < \omega_1} L_\alpha\), where \(2\text{-dim } L_\alpha < \omega_1\), for each \(\alpha < \omega_1\).
- \(L = \bigcup_{\alpha < \omega_1} L_\alpha\), where \(I\text{-dim } L_\alpha < \omega_1\), for each \(\alpha < \omega_1\).
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Theorem (K., 2019)

All prime countably saturated linear orders are isomorphic.
**Proposition**

Let $(L, \leq)$ be countably saturated linear order. The following are equivalent:

- $L$ is prime.
- $L = \bigcup_{\alpha < \omega_1} L_\alpha$, where $2\text{-dim } L_\alpha < \omega_1$, for each $\alpha < \omega_1$.
- $L = \bigcup_{\alpha < \omega_1} L_\alpha$, where $I\text{-dim } L_\alpha < \omega_1$, for each $\alpha < \omega_1$.

**Theorem (K., 2019)**

All prime countably saturated linear orders are isomorphic.

**Question**

Can we add $I\text{-dim } L = \omega_1$ to the list in previous theorem?
Thank You for attention!

References:

- M. Novotný, *On similarity of ordered continua of types $\tau$ and $\tau^2*\), Československá Akademie Věd. Časopis Pro Pěstování Matematiky, 78 (1953)

