Maximality and Resurrection

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The Maximality Principle

The maximality principle has been studied in various forms by Stavi, Väänänen, Hamkins, Fuchs, and Leibman.

Definition

Let Γ be a class of notions of forcing that is defined by some formula $\psi_\Gamma(x, p)$, where p is a parameter. In cascaded modal operator usages, $\psi_\Gamma(x, p)$ is to be evaluated in the forcing extensions.

We say that a sentence $\varphi(\vec{a})$ is Γ-*forceable* if there is $\mathbb{P} \in \Gamma$ such that for every $q \in \mathbb{P}$, we have that $q \Vdash \varphi(\vec{a})$. In other words, a sentence is Γ-forceable if it is forced to be true in an extension by a forcing from Γ.
In modal notation, write $\Diamond\varphi$.

A sentence $\varphi(\vec{a})$ is Γ-*necessary* if for all $\mathbb{P} \in \Gamma$ and all $q \in \mathbb{P}$, we have that $q \Vdash \varphi(\vec{a})$. So a sentence is Γ-necessary if it holds in any forcing extension by a forcing from Γ.
In modal notation, write $\Box\varphi$.

If S is a term in the language of set theory, then the **Maximality Principle** for Γ with parameters from S, which we denote $\mathsf{MP}_\Gamma(S)$, is the scheme of formulae stating that every sentence with parameters from S which is Γ-forceably Γ-necessary is true; i.e., the sentence "$\varphi(\vec{a})$ is Γ-necessary" is Γ-forceable, is true.
In modal notation, write: $\Diamond\Box\varphi \implies \varphi$.

Parameters?

$S = H_{\omega_1}$ is the natural parameter set for $\mathrm{MP}(S)$. Write **MP** for the boldface version of the maximality principle for all forcing, i.e., $\mathbf{MP} = \mathrm{MP}(H_{\omega_1})$.

Lemma (Fuchs)

Let Γ be a class of forcing notions which contains forcing notions which may collapse cardinals to ω_1, but no forcing in Γ may collapse a cardinal to be ω. Then $\mathrm{MP}_\Gamma(S) \implies S \subseteq H_{\omega_2}$.

Write $\mathbf{MP}_c$ for $\mathrm{MP}_{<\omega_1-closed}(H_{\omega_2})$, $\mathbf{MP}_p = \mathrm{MP}_{proper}(H_{\omega_2})$.

Lemma (Hamkins)

Let Γ be a class of forcing notions which may add an arbitrary amount of reals, but cannot collapse sizes. Then $\mathrm{MP}_\Gamma(S) \implies S \subseteq H_\mathfrak{c}$.

Thus write $\mathbf{MP}_{ccc}$ for $\mathrm{MP}_{ccc}(H_\mathfrak{c})$.

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Consistency of the maximality principle

A regular cardinal κ is **fully reflecting** so long as $V_\kappa \prec V$.

Theorem (Hamkins)

If **MP** *holds then* $\aleph_1^V$ *is fully reflecting in* L*.*

Proof.

Assume $L \models \exists z\ \varphi(z, \vec{a})$, $\vec{a} \in L_{\omega_1}$. Consider the sentence:
"The least ordinal γ such that there is $b \in L_\gamma$ with $\varphi^L(b, \vec{a})$ has cardinality at most $\aleph_0$."
By **MP** it is true, meaning that there is a witness for the existential statement in L_{ω_1}. □

Theorem (Hamkins)

Let δ *be fully reflecting. Then there are forcing extensions in which the following hold:*

- **MP** *and* $\delta = \mathfrak{c} = \aleph_1$*.*
- $\mathbf{MP}_{ccc}$ *and* $\delta = \mathfrak{c}$*.*
- $\mathbf{MP}_p$ *and* $\delta = \mathfrak{c} = \aleph_2$*.*
- $\mathbf{MP}_c$ *and* $\delta = \aleph_2$ *and* CH*.*

and so on.

Forcing Maximality from a Fully Reflecting Cardinal

Proof outline.

Define $\mathbb{P}_\kappa$, a finite support iteration, as follows:
For $\alpha < \kappa$, let Φ be the collection of sentences $\varphi(\vec{a})$ where $\vec{a} \in (H_{\omega_1})^{V_\kappa^{\mathbb{P}_\alpha}}$ and $V_\kappa^{\mathbb{P}_\alpha} \models$ "$\varphi(\vec{a})$ is forceably necessary but false." Let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha = \bigoplus_{\varphi(\vec{a})\in\Phi} \mathcal{Q}$ and $\mathcal{Q}$ is the collection of least rank posets in $V_\kappa^{\mathbb{P}_\alpha}$ forcing that $\varphi(\vec{a})$ is necessary.

Let's see why this works.
Assume $\varphi(\vec{a})$, where $\vec{a} \in H_{\omega_1}$ satisfies: $V[G] \models$ "$\varphi(\vec{a})$ is forceably necessary but false".

- Since $\mathbb{P}$ has the κ-*cc*, there has to be some stage where $\vec{a}$ appears. We may argue that there is a $\beta < \kappa$ such that there is a least rank $\mathbb{Q}$ forcing $\varphi(\vec{a})$ to be necessary in $V_\kappa[G_\beta]$, as κ is fully reflecting.
- Obtain $V[G_\beta][H][G_{tail}] = V[G]$.
- Since $\varphi(\vec{a})$ is necessary in $V_\kappa[G_\beta][H]$, we have that $\varphi(\vec{a})$ is necessary in $V[G_\beta][H]$, by elementarity.
- Thus $\varphi(\vec{a})$ is true in $V[G_\beta][H][G_{tail}]$, a contradiction. □

The Local Maximality Principle

Definition

Let Γ be a reasonable class of forcing notions, and let S be a set of parameters. Let M be a defined term for a structure to be evaluated in forcing extensions, and $S \subseteq M$.

The **Local Maximality Principle** relative to M ($\mathrm{MP}_\Gamma^M(S)$) is the statement that for every $\varphi(\vec{a})$, if $\varphi^M(\vec{a})$ is Γ-forceably Γ-necessary, then $\varphi^M(\vec{a})$ is true.

We look at $\mathbf{LMP} = \mathrm{MP}_{all}^{H_{\omega_1}}(H_{\omega_1})$, and $\mathbf{LMP}_p = \mathrm{MP}_{proper}^{H_{\omega_2}}(H_{\omega_2})$.

We have $\mathbf{MP} \implies \mathbf{LMP}$ and, for example, $\mathbf{MP}_p \implies \mathbf{LMP}_p \implies \mathrm{BPFA}$.

An inaccessible cardinal κ is **locally uplifting** so long as for every $\varphi(\vec{a})$ with $\vec{a} \in V_\kappa$, for every θ there is an inaccessible $\gamma > \theta$ such that

$$V_\kappa \models \varphi(\vec{a}) \iff V_\gamma \models \varphi(\vec{a}).$$

We have κ is fully reflecting $\implies$ κ is locally uplifting $\implies$ κ is Σ_1-reflecting.

Theorem (Consistency of Local Maximality)

- *If κ is locally uplifting, then there is a forcing extension in which* **LMP** *holds and $\kappa = \aleph_1$.*
- *If* **LMP** *holds, then $\aleph_1^V$ is locally uplifting in L.*

The Resurrection Axiom

The resurrection axiom has been studied by Hamkins and Johnstone.

Definition

Let Γ be a fixed, definable class of forcing notions.

The (lightface) **Resurrection Axiom** $\mathrm{RA}_\Gamma(H_\mathfrak{c})$ asserts that for every forcing notion $\mathbb{Q} \in \Gamma$ there is a further forcing $\dot{\mathbb{R}}$ with $\Vdash_\mathbb{Q} \dot{\mathbb{R}} \in \Gamma$ such that if $g * h \subseteq \mathbb{Q} * \dot{\mathbb{R}}$ is V-generic, then

$$H_\mathfrak{c}^V \prec H_\mathfrak{c}^{V[g*h]}.$$

The **Boldface Resurrection Axiom** $\mathbf{RA}_\Gamma(H_\mathfrak{c})$ asserts that for every forcing notion $\mathbb{Q} \in \Gamma$ and $A \subseteq H_\mathfrak{c}$ there is a further forcing $\dot{\mathbb{R}}$ with $\Vdash_\mathbb{Q} \dot{\mathbb{R}} \in \Gamma$ such that if $g * h \subseteq \mathbb{Q} * \dot{\mathbb{R}}$ is V-generic, then there is an $A^* \in V[G * h]$ such that

$$\langle H_\mathfrak{c}^V, \in, A\rangle \prec \langle H_\mathfrak{c}^{V[g*h]}, \in, A^*\rangle.$$

We consider $\mathbf{RA} = \mathbf{RA}_{all}(H_\mathfrak{c})$, $\mathbf{RA}_{ccc} = \mathbf{RA}_{ccc}(H_\mathfrak{c})$, and $\mathbf{RA}_p = \mathbf{RA}_{proper}(H_\mathfrak{c})$.

Which structures to resurrect?

Sometimes it makes sense to consider different structures than $H_{\mathfrak{c}}$ in the definition.

Lemma (Hamkins, Johnstone)

If Γ contains a forcing which forces CH but no forcing in Γ adds new reals, then $\mathrm{RA}_\Gamma(H_{\mathfrak{c}})$ is equivalent to CH.

Proposition

Suppose Γ contains forcing to collapse to ω_1 and no forcing in Γ adds new reals. Then $\mathrm{RA}_\Gamma(H_{2^{\aleph_1}}) \iff 2^{\aleph_1} = \aleph_2 + \mathrm{RA}_\Gamma(H_{\omega_2})$.

We consider $\mathbf{RA}_c = \mathbf{RA}_c(H_{\omega_2})$.

Consistency of the Resurrection Axiom

An inaccessible cardinal κ is **uplifting** so long as for every ordinal θ there is an inaccessible $\gamma \geq \theta$ such that $V_\kappa \prec V_\gamma$ is a proper elementary extension.

We say that κ is **strongly uplifting** if it is strongly θ-uplifting if for every $A \subseteq V_\kappa$ there is an inaccessible $\gamma \geq \theta$ and a set $A^* \subseteq V_\gamma$ such that $\langle V_\kappa, \in, A\rangle \prec \langle V_\gamma, \in, A^*\rangle$ is a proper elementary extension.

Note κ is strongly uplifting $\implies$ κ is uplifting $\implies$ κ is locally uplifting $\implies$ κ is Σ_1-reflecting.

Theorem (Hamkins, Johnstone)

- *If* **RA** *holds then* $\mathfrak{c}^V = \aleph_1^V$ *is strongly uplifting in* L.
- *Let* κ *be strongly uplifting. Then there are forcing extensions in which we have the following:*
  - **RA** *and* $\kappa = \mathfrak{c} = \aleph_1$.
  - $\mathbf{RA}_{ccc}$ *and* $\kappa = \mathfrak{c}$.
  - $\mathbf{RA}_p$ *and* $\kappa = \mathfrak{c} = \aleph_2$.
  - $\mathbf{RA}_c$ *and* $\kappa = \aleph_2$ *and* CH.

  and so on.

Thus **RA** $\implies$ RA $\implies$ **LMP**, and we have, e.g.: $\mathbf{RA}_p \implies \mathrm{RA}_p \implies \mathbf{LMP}_p \implies$ BPFA.

Resurrection's equiconsistency with the existence of a strongly uplifting cardinal

Proof sketch.

Let **RA** hold, and let $\kappa = \mathfrak{c}^V = \aleph_1^V$. Fix any cardinal $\theta > \kappa$, and consider $\mathcal{C}oll(\omega, \theta)$. There is a further forcing such that $\langle H_{\mathfrak{c}}^V, \in, A \rangle \prec \langle H_{\mathfrak{c}}^{V[g*h]}, \in, A^* \rangle$. Let $\gamma = \mathfrak{c}^{V[g*h]}$. It follows that $\aleph_1^{V[g*h]} = \gamma$ and $\gamma > \theta$ and $\langle H_\kappa^L, \in, A \rangle \prec \langle H_\gamma^L, \in, A^* \rangle$, so κ is strongly uplifting in L.

Let κ be strongly uplifting. Define $\mathbb{P}_\kappa$, a finite support iteration, as follows:
For $\alpha < \kappa$, let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ such that $\dot{\mathbb{Q}}_\alpha = \oplus \mathcal{Q}$ where $\mathcal{Q}$ is the collection of least rank posets in $V_\kappa^{\mathbb{P}_\alpha}$ for which resurrection fails.

Suppose toward a contradiction that **RA** fails in $V[G]$ as witnessed by $\mathbb{Q}$ of least rank.

- Use the uplifting property of κ to argue that $\mathbb{Q}$ appears at stage κ of the exact iteration defined in some large enough inaccessible γ to obtain $\mathbb{P}_\gamma = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \mathbb{P}_{tail}$.
- Lift the strongly uplifting embedding to $\langle H_\kappa[G_\kappa], \in, \mathbb{P}, \dot{A} \rangle \prec \langle H_\gamma[G_\gamma], \in, \mathbb{P}_\gamma, \dot{A}^* \rangle$.
- Thus $\langle H_{\mathfrak{c}}^{V[G]}, \in, A \rangle \prec \langle H_{\mathfrak{c}}^{V[G_\gamma]}, \in A^* \rangle$, a contradiction to $\mathbb{Q}$ being a counterexample. □

Maximality vs. Resurrection

So both **MP** and **RA** imply **LMP**. Do the two simply imply each other?

$\neg$(**MP** $\implies$ **RA**)

If κ is fully reflecting, take the least γ such that $V_\kappa \prec V_\gamma$. If there isn't such a γ, then κ isn't uplifting anyway.

Then in V_γ, we have that κ is not even uplifting.

$\neg$(**RA** $\implies$ **MP**)

Working in a minimal model of $T = \text{ZFC} + \text{“}V = L\text{”} + \text{“there is a strongly uplifting cardinal”}$ (i.e., no initial segment of the model satisfies this theory), we may force to obtain **RA**.

Now **MP** can't hold in the extension, since letting κ be the $\aleph_1$ of the extension, L_κ is elementary in L.

Combining Maximality and Resurrection

An inaccessible cardinal κ is **strongly uplifting fully reflecting** so long as:

- κ is fully reflecting, i.e. $V_\kappa \prec V$
- κ is strongly uplifting

If there is a subtle cardinal, then it is consistent that there is a strongly uplifting fully reflecting cardinal.

Theorem

If both **RA** *and* **MP** *both hold, then* $\mathfrak{c}^V$ *is strongly uplifting fully reflecting in L.*

Theorem

Let κ be a strongly uplifting fully reflecting cardinal. Then there are forcing extensions in which we have the following:

- $\mathbf{RA} + \mathbf{MP} + \kappa = \mathfrak{c} = \aleph_1$.
- $\mathbf{RA}_{ccc} + \mathbf{MP}_{ccc} + \kappa = \mathfrak{c}$.
- $\mathbf{RA}_p + \mathbf{MP}_p + \kappa = \mathfrak{c} = \aleph_2$.
- $\mathbf{RA}_c + \mathbf{MP}_c + \kappa = \aleph_2 + \mathrm{CH}$.

and so on.

Forcing MP + RA

Proof idea.

Let κ be strongly uplifting fully reflecting. Define $\mathbb{P}$ as a finite support iteration of length κ so that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ where $\dot{\mathbb{Q}}_\alpha$ is a term for the lottery sum

$$\oplus\mathcal{R} \bigoplus \oplus\mathcal{M},$$

where $\mathcal{R}$ is the collection of least-rank counterexamples to boldface resurrection, and $\mathcal{M}$ is the collection of least-rank counterexamples to the maximality principle (defined as in those iterations). □

Thank you.