PAVOL JOZEF ŠAFÁRIK UNIVERSITY IN KOŠICE
Faculty of Science

INSTITUTE OF MATHEMATICS

The first sketch: $\mathcal{N}$ with respect to ideals on ω

Viera Šottová

joint work with Diego A. Mejía

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Basic notions

- The family $\mathcal{J} \subseteq \mathcal{P}(\omega)$ is called **ideal** on ω, if
  - it is closed under taking subsets and finite unions
  - does not contain the set ω, but contains all finite subsets of ω.
- Examples:
  - the Frechét ideal, denoted as Fin, is a set $[\omega]^{<\aleph_0}$,
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- Examples:
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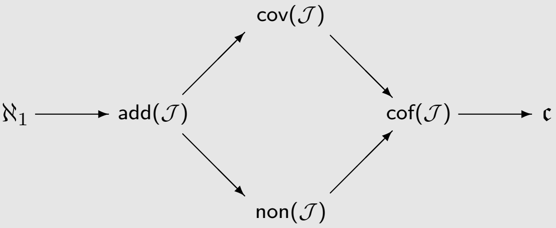
  - $\mathcal{Z}$ is an asymptotic density zero ideal,
  - nwd is nowhere dense ideal.
- Ideal $\mathcal{J}$ a is σ-ideal if it is an ideal closed under σ unions.
  - ideal of meager sets $\mathcal{M}$,
  - ideal of Lebesgue measure zero sets $\mathcal{N}$.
- Note: such ideals are usually defined on $\mathbb{R}$.

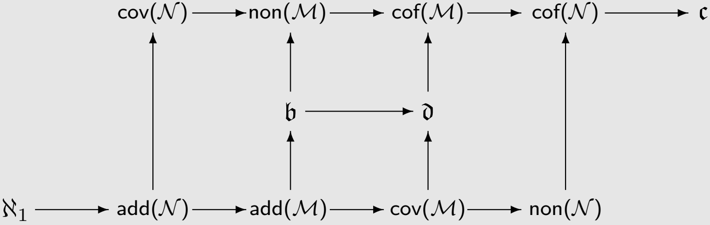
Basic notions

- For any ideal $\mathcal{J}$ we can consider cardinal invariants
  - $\mathtt{add}(\mathcal{J}) = \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ \bigcup \mathcal{A} \notin \mathcal{J}\}$,
  - $\mathtt{cov}(\mathcal{J}) = \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ \bigcup \mathcal{A} = X\}$,
  - $\mathtt{non}(\mathcal{J}) = \{|Y| : Y \subseteq X \ \& \ Y \notin \mathcal{J}\}$,
  - $\mathtt{cof}(\mathcal{J}) = \{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ (\forall B \in \mathcal{J})(\exists A \in \mathcal{A}) B \subseteq A\}$.

- For a σ-ideal $\mathcal{J}$



Cichoń's diagram.



- $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\}$ and $\text{cof}(\mathcal{M}) = \max\{\text{non}(\mathcal{M}), \mathfrak{d}\}$.

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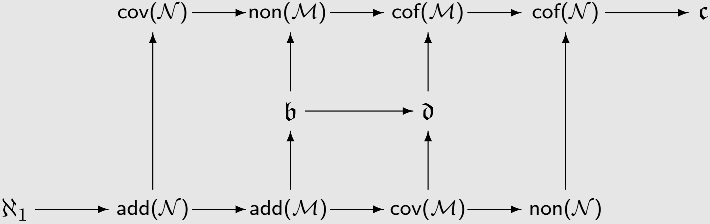

- $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$.
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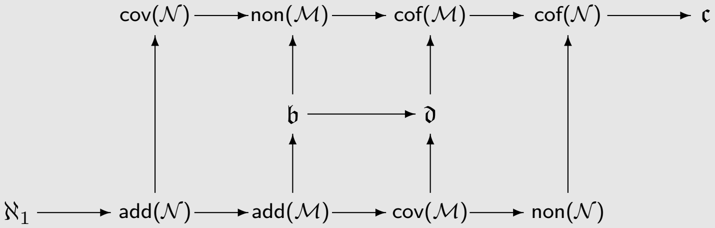

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- $\mathfrak{b}$ is a bounding number,
- $\mathfrak{d}$ is a dominating number.
- They both can be redefined via ideals on ω i.e., $\mathfrak{b}_{\mathcal{J}}$ and $\mathfrak{d}_{\mathcal{J}}$.

Combinatorial characterization of $\mathcal{N}$

- Let $\bar{c}$ be a sequence $\langle c_n : \ n \in \omega \rangle$.

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- By T. Bartoszyński:

Let $X \subseteq 2^\omega$. Then

$$X \in \mathcal{N} \ \Leftrightarrow \ (\exists \bar{c} \in \Omega^*) \ X \subseteq N(\bar{c})$$

where $N(\bar{c}) = \{x \in 2^\omega : \ |\{n \in \omega : \ x \in c_n\}| = \aleph_0\}$.

- Moreover, $N(\bar{c}) = \bigcap_{n<\omega} \bigcup_{m \geq n} c_m$.

From $\mathcal{N}$ to $\mathcal{N}_{\mathcal{J}}$

- Consider a set

$$\Omega^{**} = \left\{ \bar{c} : \; (\forall n \in \omega) \; c_n \text{ is open and } \sum_{n<\omega} \mu(c_n) < \infty \right\}.$$

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- Define following sets for an ideal $\mathcal{J}$ on ω and for each $\bar{c} \in \Omega^{**}$

$$N_{\mathcal{J}}(\bar{c}) = \left\{ x \in 2^{\omega} : \ \{n \in \omega : \ x \in c_n\} \in \mathcal{J}^{+} \right\}.$$

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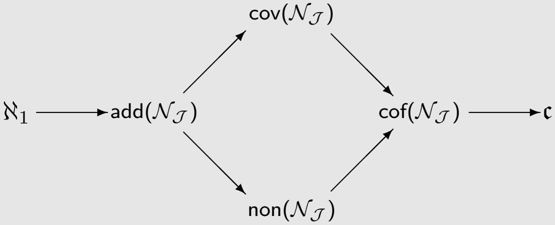
- Then

$$\mathcal{N}_{\mathcal{J}} = \{ X \subseteq 2^{\omega} : \ (\exists \bar{c} \in \Omega^{**}) \ X \subseteq N_{\mathcal{J}}(\bar{c}) \}.$$

Basic properties of $\mathcal{N}_\mathcal{J}$

- $\mathcal{N}_\mathcal{J}$ is a σ-ideal for any ideal $\mathcal{J}$ on ω,



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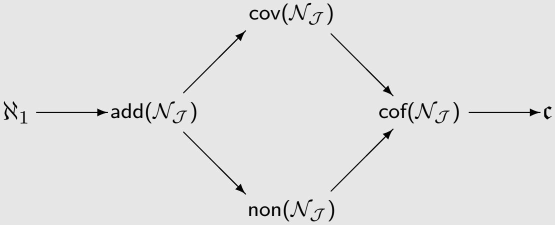

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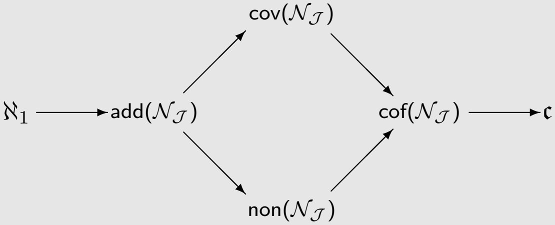

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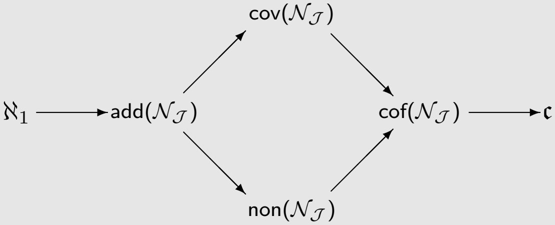

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Observation

Let $\mathcal{J}$ be an ideal on ω which has the Baire property[a]*. Then $\mathcal{N} = \mathcal{N}_{\mathcal{J}}$.*

[a] $\mathcal{J}$ has Baire property if and only if $\mathcal{J}$ is meager if and only if $\mathrm{Fin} \leq_{RB} \mathcal{J}$.

Relations of cardinal invariants for $\mathcal{N}$ and $\mathcal{N}_{\mathcal{J}}$

Proposition (D. A. Mejía, V. Š.)

Let $\mathcal{J}$ *be an arbitrary ideal on* ω. *Then*

- $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{N}_{\mathcal{J}})$,
- $\mathrm{cof}(\mathcal{N}_{\mathcal{J}}) \leq \mathrm{cof}(\mathcal{N})$.

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- Note that if $\mathcal{I} \subseteq \mathcal{J}$ then $\mathtt{cov}(\mathcal{I}) \leq \mathtt{cov}(\mathcal{J})$ and $\mathtt{non}(\mathcal{J}) \leq \mathtt{non}(\mathcal{I})$.

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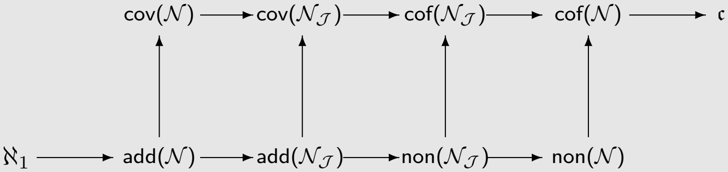

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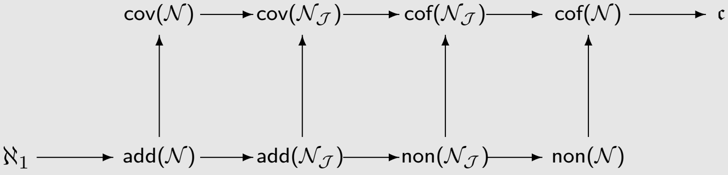

- The consistency of $\mathtt{cov}(\mathcal{N}) < \mathtt{non}(\mathcal{N}) \Rightarrow$ the consistency of $\mathtt{cov}(\mathcal{N}_{\mathcal{J}}) < \mathtt{non}(\mathcal{N}_{\mathcal{J}})$.

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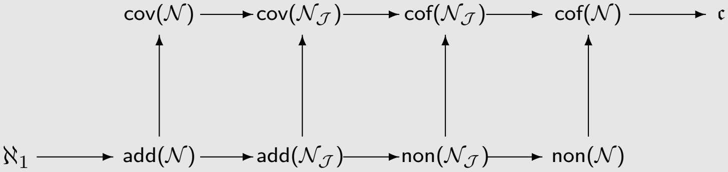

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- Is $\mathtt{non}(\mathcal{N}_{\mathcal{J}}) < \mathtt{cov}(\mathcal{N}_{\mathcal{J}})$ consistent in $\mathbf{ZFC}$?

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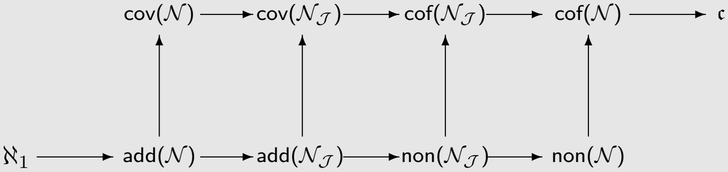

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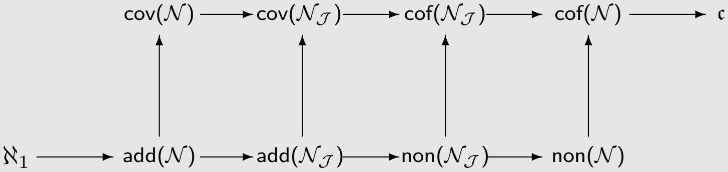

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  - If $Z \in \mathcal{N}_{\mathcal{J}}$ and $x \in 2^{\omega}$ then $x + Z \in \mathcal{N}_{\mathcal{J}}$.

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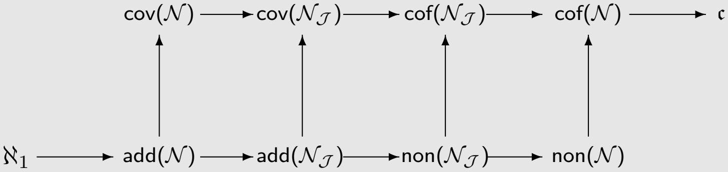

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  - If $Z \in \mathcal{N}_{\mathcal{J}}$ and $x \in 2^{\omega}$ then $x + Z \in \mathcal{N}_{\mathcal{J}}$.
  - Is there $Z \in \mathcal{N}_{\mathcal{J}}$ such that $(2^{\omega} \setminus Z) \in \mathcal{M}$?

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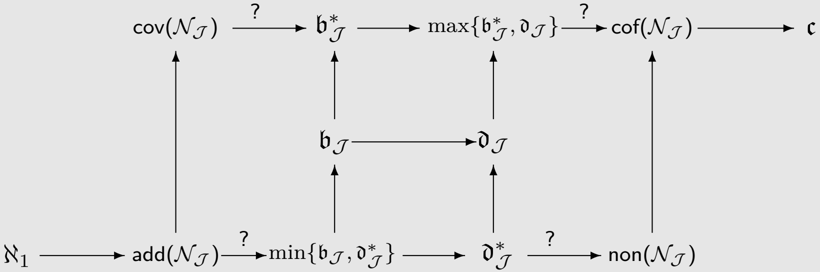
- by T. Bartoszyński
  - $\mathtt{non}(\mathcal{M}) = \mathfrak{b}^*_{\mathrm{Fin}} = \min\{|F| : F \subset \omega^\omega, \neg(\exists y \in \omega^\omega)(\forall x \in F) x \neq^* y\}$,
  - $\mathtt{cov}(\mathcal{M}) = \mathfrak{d}^*_{\mathrm{Fin}} = \min\{|D| : D \subset \omega^\omega, (\forall x \in \omega^\omega)(\exists y \in D) x \neq^* y\}$,

  where $x \neq^* y$ iff $\{n : x(n) = y(n)\} \in \mathrm{Fin}$.
- we define $\mathfrak{b}^*_{\mathcal{J}}$ and $\mathfrak{d}^*_{\mathcal{J}}$ in a case if $\{n : x(n) = y(n)\} \in \mathcal{J}$.

The last idea



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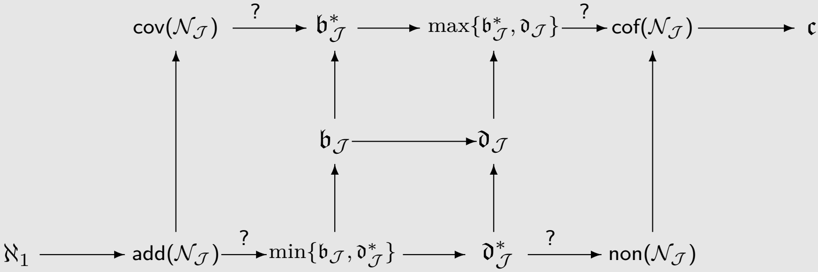

Observation

If $\mathcal{J}$ has the Baire property, then $\mathfrak{b}^*_{\mathcal{J}} = \mathfrak{b}^*_{\mathrm{Fin}}$ and $\mathfrak{d}^*_{\mathcal{J}} = \mathfrak{d}^*_{\mathrm{Fin}}$.

Thank you for your attention

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