Small almost disjoint families with applications

Grzegorz Plebanek

University of Wrocław

Winter School in Abstract Analysis, Hejnice (2019)

joint work with

Antonio Avilés and **Witold Marciszewski**

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For $n \geq 2$ we write $\mathfrak{a}_n$ for the minimal size of an almost disjoint family $\mathscr{A}$ that can be divided into disjoint parts $\mathscr{A}_1, \ldots, \mathscr{A}_n$ that are not separated.
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$\mathfrak{a}_\omega \leq \mathrm{non}(\mathscr{E})$, where $\mathscr{E}$ is the σ-ideal of subsets of 2^ω that can be covered by countably many closed measure zero sets.

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Bartoszyński & Shelah: Consistently,

$$\mathrm{non}(\mathscr{E}) < \min(\mathrm{non}(\mathscr{N}), \mathrm{non}(\mathscr{M}))$$

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Then $\mathscr{A} = \mathscr{A}_0 \cup \ldots \cup \mathscr{A}_{n-1}$ is an almost disjoint family and $\mathscr{A}_i$ are not separated.

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- If K is metrizable then there is a norm-one extension operator for every compact $L \supseteq K$ (Borsuk-Dugundji).
- If K is not *ccc* and $L \supseteq K$ is separable then there is no extension operator (Pełczyński).

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Corollary

If K satisifes the assumptions of Proposition for every n then K has a countable discrete extension without extension operators.

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