Indestructibility of the tree property

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Recall the following definition:

**Definition**

Let $\kappa$ be a regular cardinal. A $\kappa$-tree is called *Aronszajn* if it has no cofinal branch. The tree property holds at $\kappa$ if there are no $\kappa$-Aronszajn trees.
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The tree property is a compactness property which has been recently extensively studied. In order to construct models with the tree property, it is helpful to try to understand which forcings cannot create new $\kappa$-Aronszajn trees (we say that the tree property is *indestructible* by these forcings).
In this talk, we will discuss primarily the indestructibility over the Mitchell model $V[\mathcal{M}(\omega, \kappa)]$ in which the tree property holds at $\omega_2 = \kappa$ with $2^\omega = \omega_2$. We assume that $\kappa$ is supercompact in $V$. 
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Let us write $M = M(\omega, \kappa)$. We will not define $M$. For our purposes it suffices to say that there are projections

$$\text{Add}(\omega, \kappa) \times T \rightarrow_{\text{onto}} M \rightarrow_{\text{onto}} \text{Add}(\omega, \kappa)$$

for some $\omega_1$-closed forcing $T$ (the term forcing),

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for some \( \omega_1 \)-closed forcing \( T \) (the term forcing), and that if \( \alpha < \kappa \) is inaccessible, then in \( V[\mathbb{M}_\alpha] \) there are projections

\[
\text{Add}(\omega, \kappa) \times T_\alpha \rightarrow_{\text{onto}} \mathbb{M}/\mathbb{M}_\alpha \rightarrow_{\text{onto}} \text{Add}(\omega, \kappa),
\]

where \( \mathbb{M}_\alpha \) is regularly embedded into \( \mathbb{M} \) as its “initial segment” and \( T_\alpha \) is \( \omega_1 \)-closed in \( V[\mathbb{M}_\alpha] \).
The product analysis on the previous slide and the so called branch lemmas imply (using a standard method which we will omit here) the tree property at $\omega_2$. For clarity we state the lemmas just for the specific case of $\omega_2$. 
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- **(Folkore)** Assume that $P$ is a ccc forcing notion. Then $P$ does not add cofinal branches to $\omega_2$-Aronszajn trees.

- **(Unger)** Suppose $2^{\omega} \geq \omega_2$. Assume that $P$ and $Q$ are forcing notions such that $P$ is ccc and $Q$ is $\omega_1$-closed. If $T$ is a $\omega_2$-tree in $V[P]$, then forcing with $Q$ over $V[P]$ does not add cofinal branches to $T$. 
Remark. Earlier arguments for the tree property were based on Silver’s lemma\(^1\) and require in our specific case a stronger property of \(\omega_1\)-square-cc for \(P\).\(^2\) Unger’s result strengthens Silver’s lemma and consequently weakens the assumption required for \(P\) to ccc. This is not important for Add(\(\omega, \kappa\)) (which is even \(\omega_1\)-Knaster) but becomes important if we wish to consider arbitrary ccc forcings for the indestructibility results.

\(^1\)2\(\omega \geq \omega_2\) implies that \(\omega_1\)-closed forcings do not add cofinal branches to \(\omega_2\)-trees
\(^2\)\(P\) is \(\omega_1\)-square-cc iff \(P \times P\) is ccc
Let us discuss the following result:

**Theorem (Honzik, S. (2019))**

Suppose $\kappa$ is supercompact. The tree property at $\omega_2$ in $V[M]$ is indestructible by all ccc forcings which live in $V[\text{Add}(\omega, \kappa)]$. 

First notice that $V[\text{Add}(\omega, \kappa)] \subseteq V[M]$ so the statement of the theorem makes sense.

Let us repeat that we require just ccc and not a stronger form of ccc such as $\omega_1$-Knaster or "square-ccc", which is often required.
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First notice that $V[\text{Add}(\omega, \kappa)] \subseteq V[M]$ so the statement of the theorem makes sense.

Let us repeat that we require just ccc and not a stronger form of ccc such as $\omega_1$-Knaster or “square-ccc”, which is often required.
Suppose $\mathbb{Q}$ is a ccc forcing in $V[\text{Add}(\omega, \kappa)]$ and let us fix an $\text{Add}(\omega, \kappa)$-name $\check{\mathbb{Q}}$ for $\mathbb{Q}$. 
Suppose \( \mathbb{Q} \) is a ccc forcing in \( V[\mathrm{Add}(\omega, \kappa)] \) and let us fix an \( \mathrm{Add}(\omega, \kappa) \)-name \( \dot{\mathbb{Q}} \) for \( \mathbb{Q} \).

Choose a supercompact embedding \( j : V \rightarrow M \) with critical point \( \kappa \) so that \( M \) is closed under sequences of size \( |M * \dot{\mathbb{Q}}| \).
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- Choose a supercompact embedding \( j : V \rightarrow M \) with critical point \( \kappa \) so that \( M \) is closed under sequences of size \( |M \ast \check{\mathbb{Q}}| \).
- \( j \) restricted to \( M \ast \check{\mathbb{Q}} \) is a regular embedding into \( j(M \ast \check{\mathbb{Q}}) \) due to \( M \ast \check{\mathbb{Q}} \) being \( \kappa \)-cc, and one can therefore lift to \( j : V[M \ast \check{\mathbb{Q}}] \rightarrow M[j(M \ast \check{\mathbb{Q}})] \).
Suppose $Q$ is a ccc forcing in $V[\text{Add}(\omega, \kappa)]$ and let us fix an Add($\omega, \kappa$)-name $\dot{Q}$ for $Q$.

- Choose a supercompact embedding $j : V \to M$ with critical point $\kappa$ so that $M$ is closed under sequences of size $|M*\dot{Q}|$.
- $j$ restricted to $M*\dot{Q}$ is a regular embedding into $j(M*\dot{Q})$ due to $M*\dot{Q}$ being $\kappa$-cc, and one can therefore lift to $j : V[M*\dot{Q}] \to M[j(M*\dot{Q})]$.
- Since $M$ is closed under sequences of size $|M*\dot{Q}|$, the regular embedding is an element of $M$, and it follows that $M[j(M*\dot{Q})]$ can be written as $M[M*\dot{Q} * \dot{Q}]$ for some forcing $\dot{Q}$. 
Over $M[\mathbb{M} \ast \dot{Q}]$ there is a projection from the product

$$j(\text{Add}(\omega, \kappa) \ast \dot{Q})/(\text{Add}(\omega, \kappa) \ast \dot{Q}) \times T_\kappa$$

onto $\dot{Q}$, where the first component of the product is ccc in $M[\mathbb{M} \ast \dot{Q}]$ and $T_\kappa$ is $\omega_1$-closed in $M[\mathbb{M}]$. 
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The argument can be finished by the standard method using the fact that a ccc forcing cannot add a cofinal branch to an $\omega_2$-Aronszajn tree, and neither can an $\omega_1$-closed forcing over a ccc forcing.
The theorem can be generalized by using a more complex forcing $R$ (with similar properties as $M$) which yields the following:

**Theorem (Honzik, S. (2019))**

*Suppose $\kappa$ is supercompact. There is a forcing $R$ such that over $V[R]$ the tree property at $\omega_2 = \kappa$ is indestructible by all these forcings:*

- $\text{ccc forcings living in } V[\text{Add}(\omega, \kappa)]$.
- $\omega_1$-closed, $\omega_2$-cc forcings.
- $\omega_2$-directed closed forcings.
- $\omega_1$-cc or $\omega_1$-distributive forcings of size $\omega_1$. 
Some applications. The theorem generalizes to other cardinals: one can show that if $\kappa < \lambda$ are regular and $\lambda$ supercompact, then all $\kappa^+-\text{cc}$ forcings living in $V[\text{Add}(\kappa, \lambda)]$ preserve the tree property at $\lambda = \kappa^{++}V[M(\kappa, \lambda)]$. With the additional assumption that $\kappa$ is Laver-indestructible supercompact:
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- $\mathbb{M}(\kappa, \lambda) \ast \text{Prk}^{V[\text{Add}(\kappa, \lambda)]}$ forces the tree property at the double successor of a singular strong limit cardinal with countable cofinality, where $\text{Prk}^M$ denotes the vanilla Prikry forcing defined in a model $M$. 
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- $\mathcal{M}(\kappa, \lambda) \ast \text{Prk}^V[\text{Add}(\kappa, \lambda)]$ forces the tree property at the double successor of a singular strong limit cardinal with countable cofinality, where $\text{Prk}^M$ denotes the vanilla Prikry forcing defined in a model $M$.

- $\mathcal{M}(\kappa, \lambda) \ast (\text{Add}(\kappa, \delta) \ast \text{Prk}^V[\text{Add}(\kappa, \delta)])$ forces the tree property at the double successor of a singular strong limit cardinal with countable cofinality, while making $2^\kappa$ arbitrarily large (by choosing a large enough $\delta$).
\( \mathbb{M}(\kappa, \lambda) \ast (\text{Add}(\kappa, \delta) \ast \text{Mag}^V[\text{Add}(\kappa, \delta)]) \) forces the tree property at the double successor of a singular strong limit cardinal with an arbitrary cofinality \( \omega < \mu < \kappa \), while making \( 2^\kappa \) arbitrarily large, where \( \text{Mag}^M \) is the Magidor forcing in \( M \) defined with respect to a sequence of measures of length \( \mu \).
M(\kappa, \lambda) \ast (\text{Add}(\kappa, \delta) \ast \text{Mag}^V[\text{Add}(\kappa, \delta)]) forces the tree property at the double successor of a singular strong limit cardinal with an arbitrary cofinality \omega < \mu < \kappa, while making \(2^\kappa\) arbitrarily large, where \text{Mag}^M is the Magidor forcing in \text{M} defined with respect to a sequence of measures of length \mu.

There are applications for cardinal invariants (for instance to the ultrafilter number on \kappa because all subsets of \kappa added by \text{M}(\kappa, \lambda) are added already by \text{Add}(\kappa, \lambda)).
Open questions. Let us mention just one important question:

Q1. Is the tree property indestructible over $V[\mathcal{M}]$ by all ccc forcings $Q$ living in $V[\mathcal{M}]$? Or more generally, by all $\kappa^+-cc$ forcings $Q$ living in $V[\mathcal{M}(\kappa, \lambda)]$ ($\kappa$ can be measurable now)?
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Notice that our proof relies heavily on the product analysis which uses the fact that $\text{Add}(\kappa, \lambda) * \dot{\mathcal{Q}}$ is meaningful (and $\kappa^+-\text{cc}$). This cannot be done if $\mathcal{Q}$ lives in $V[\mathcal{M}(\kappa, \lambda)]$. 