Indestructibility of the tree property

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The tree property is a compactness property which has been recently extensively studied. In order to construct models with the tree property, it is helpful to try to understand which forcings cannot create new κ-Aronszajn trees (we say that the tree property is *indestructible* by these forcings).

In this talk, we will discuss primarily the indestructibility over the Mitchell model $V[\mathbb{M}(\omega, \kappa)]$ in which the tree property holds at $\omega_2 = \kappa$ with $2^\omega = \omega_2$. We assume that κ is supercompact in V.

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Let us write $\mathbb{M} = \mathbb{M}(\omega, \kappa)$. We will not define $\mathbb{M}$. For our purposes it suffices to say that there are projections

$$\mathrm{Add}(\omega, \kappa) \times \mathbb{T} \to_{\mathrm{onto}} \mathbb{M} \to_{\mathrm{onto}} \mathrm{Add}(\omega, \kappa)$$

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for some ω_1-closed forcing $\mathbb{T}$ (the term forcing),and that if $\alpha < \kappa$ is inaccessible, then in $V[\mathbb{M}_\alpha]$ there are projections

$$\mathrm{Add}(\omega,\kappa) \times \mathbb{T}_\alpha \to_{\mathrm{onto}} \mathbb{M}/\mathbb{M}_\alpha \to_{\mathrm{onto}} \mathrm{Add}(\omega,\kappa),$$

where $\mathbb{M}_\alpha$ is regularly embedded into $\mathbb{M}$ as its "initial segment" and $\mathbb{T}_\alpha$ is ω_1-closed in $V[\mathbb{M}_\alpha]$.

Branch lemmas

The product analysis on the previous slide and the so called branch lemmas imply (using a standard method which we will omit here) the tree property at ω_2. For clarity we state the lemmas just for the specific case of ω_2.

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The product analysis on the previous slide and the so called branch lemmas imply (using a standard method which we will omit here) the tree property at ω_2. For clarity we state the lemmas just for the specific case of ω_2.

- (Folkore) Assume that P is a **ccc** forcing notion. Then P does not add cofinal branches to ω_2-Aronszajn trees.
- (Unger) Suppose $2^\omega \geq \omega_2$. Assume that P and Q are forcing notions such that P is **ccc** and Q is ω_1-**closed**. If T is a ω_2-tree in $V[P]$, then forcing with Q over $V[P]$ does not add cofinal branches to T.

Remark. Earlier arguments for the tree property were based on Silver's lemma[1] and require in our specific case a stronger property of ω_1-square-cc for P.[2] Unger's result strengthens Silver's lemma and consequently weakens the assumption required for P to ccc. This is not important for $\mathrm{Add}(\omega, \kappa)$ (which is even ω_1-Knaster) but becomes important if we wish to consider arbitrary ccc forcings for the indestructibility results.

[1] $2^\omega \geq \omega_2$ implies that ω_1-closed forcings do not add cofinal branches to ω_2-trees

[2] P is ω_1-square-cc iff $P \times P$ is ccc

Let us discuss the following result:

Theorem (Honzik, S. (2019))

Suppose κ is supercompact. The tree property at ω_2 in $V[\mathbb{M}]$ is indestructible by all ccc forcings which live in $V[\mathrm{Add}(\omega, \kappa)]$.

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Let us repeat that we require just ccc and not a stronger form of ccc such as ω_1-Knaster or "square-ccc", which is often required.

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- Choose a supercompact embedding $j : V \to M$ with critical point κ so that M is closed under sequences of size $|\mathbb{M} * \dot{\mathbb{Q}}|$.

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- j restricted to $\mathbb{M} * \dot{\mathbb{Q}}$ is a regular embedding into $j(\mathbb{M} * \dot{\mathbb{Q}})$ due to $\mathbb{M} * \dot{\mathbb{Q}}$ being κ-cc, and one can therefore lift to $j : V[\mathbb{M} * \dot{\mathbb{Q}}] \to M[j(\mathbb{M} * \dot{\mathbb{Q}})]$.

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- Since M is closed under sequences of size $|\mathbb{M} * \dot{\mathbb{Q}}|$, the regular embedding is an element of M, and it follows that $M[j(\mathbb{M} * \dot{\mathbb{Q}})]$ can be written as $M[\mathbb{M} * \dot{\mathbb{Q}} * \dot{Q}]$ for some forcing $\dot{Q}$.

- Over $M[\mathbb{M} * \dot{\mathbb{Q}}]$ there is a projection from the product

$$j(\mathrm{Add}(\omega, \kappa) * \dot{\mathbb{Q}})/(\mathrm{Add}(\omega, \kappa) * \dot{\mathbb{Q}}) \;\times\; \mathbb{T}_\kappa$$

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  onto $\dot{Q}$, where the first component of the product is ccc in $M[\mathbb{M} * \dot{\mathbb{Q}}]$ and $\mathbb{T}_\kappa$ is ω_1-closed in $M[\mathbb{M}]$.
- The argument can be finished by the standard method using the fact that a ccc forcing cannot add a cofinal branch to an ω_2-Aronszajn tree, and neither can an ω_1-closed forcing over a ccc forcing.

The theorem can be generalized by using a more complex forcing $\mathbb{R}$ (with similar properties as $\mathbb{M}$) which yields the following:

Theorem (Honzik, S. (2019))

Suppose κ is supercompact. There is a forcing $\mathbb{R}$ such that over $V[\mathbb{R}]$ the tree property at $\omega_2 = \kappa$ is indestructible by all these forcings:

- *ccc forcings living in $V[\mathrm{Add}(\omega, \kappa)]$.*
- *ω_1-closed, ω_2-cc forcings.*
- *ω_2-directed closed forcings.*
- *ω_1-cc or ω_1-distrubutive forcings of size ω_1.*

Some applications. The theorem generalizes to other cardinals: one can show that if $\kappa < \lambda$ are regular and λ supercompact, then all κ^+-cc forcings living in $V[\mathrm{Add}(\kappa, \lambda)]$ preserve the tree property at $\lambda = \kappa^{++V[\mathbb{M}(\kappa,\lambda)]}$. With the additional assumption that κ is Laver-indestructible supercompact:

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- $\mathbb{M}(\kappa, \lambda) * \mathrm{Prk}^{V[\mathrm{Add}(\kappa,\lambda)]}$ forces the tree property at the double successor of a singular strong limit cardinal with countable cofinality, where Prk^M denotes the vanilla Prikry forcing defined in a model M.

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- $\mathbb{M}(\kappa, \lambda) * (\mathsf{Add}(\kappa, \delta) * \mathsf{Prk}^{V[\mathsf{Add}(\kappa,\delta)]})$ forces the tree property at the double successor of a singular strong limit cardinal with countable cofinality, while making 2^κ arbitrarily large (by choosing a large enough δ).

- $\mathbb{M}(\kappa, \lambda) * (\mathsf{Add}(\kappa, \delta) * \mathsf{Mag}^{V[\mathsf{Add}(\kappa,\delta)]})$ forces the tree property at the double successor of a singular strong limit cardinal with an arbitrary cofinality $\omega < \mu < \kappa$, while making 2^κ arbitrarily large, where Mag^M is the Magidor forcing in M defined with respect to a sequence of measures of length μ.

- $\mathbb{M}(\kappa,\lambda) * (\mathsf{Add}(\kappa,\delta) * \mathsf{Mag}^{V[\mathsf{Add}(\kappa,\delta)]})$ forces the tree property at the double successor of a singular strong limit cardinal with an arbitrary cofinality $\omega < \mu < \kappa$, while making 2^κ arbitrarily large, where Mag^M is the Magidor forcing in M defined with respect to a sequence of measures of length μ.
- There are applications for cardinal invariants (for instance to the ultrafilter number on κ because all subsets of κ added by $\mathbb{M}(\kappa,\lambda)$ are added already by $\mathsf{Add}(\kappa,\lambda)$).

Open questions. Let us mention just one important question:

Q1. Is the tree property indestructible over $V[\mathbb{M}]$ by all ccc forcings $\mathbb{Q}$ living in $V[\mathbb{M}]$? Or more generally, by all κ^+-cc forcings $\mathbb{Q}$ living in $V[\mathbb{M}(\kappa, \lambda)]$ (κ can be measurable now)?

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Notice that our proof relies heavily on the product analysis which uses the fact that $\mathrm{Add}(\kappa,\lambda) * \dot{\mathbb{Q}}$ is meaningful (and κ^+-cc). This cannot be done if $\mathbb{Q}$ lives in $V[\mathbb{M}(\kappa,\lambda)]$.