SET-THEORETIC METHODS IN TOPOLOGY AND REAL FUNCTIONS THEORY
DEDICATED TO 80TH BIRTHDAY OF LEV BUKOVSKÝ

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On a level measure

Jaroslav Šupina

joint work with J. Borzová, L. Halčinová, O. Hutník and J. Kiselák

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Basic setting

\( X \) is a topological space.

\( \mathcal{E}_B \) is a family of all Borel subsets of \( X \).
Warning!

\[
\text{measure} \neq \text{measure}
\]

\[
\text{measure} = \text{function } m : \mathcal{E}_B \rightarrow [0, +\infty] \text{ such that } m(\emptyset) = 0
\]

monotone measure

non-additive integrals
measure $\neq$ measure

measure $= \text{ function } m : E_B \to [0, +\infty]$ such that $m(\emptyset) = 0$

monotone measure

non-additive integrals
Warning!

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\text{monotone measure}

\text{non-additive integrals}
Warning!

measure ≠ measure

measure = function \( m : \mathcal{E}_B \rightarrow [0, +\infty] \) such that \( m(\emptyset) = 0 \)

monotone measure

non-additive integrals
Level sets and level measure

\[ \alpha \]

\[ \{x \in X; f(x) \geq \alpha \} \]

\[ h_{\mu,f}(\alpha) = \mu(f \geq \alpha) := \mu(\{x \in X; f(x) \geq \alpha\}) \]

Integrals

- The Lebesgue integral
  \[ (L) \int_X f \ d\mu = \int_0^\infty \mu(f \geq \alpha) \ d\alpha \]

- The Choquet integral
  \[ (Ch) \int_X f \ d\mu = \int_0^\infty \mu(f \geq \alpha) \ d\alpha \]

- The Sugeno integral
  \[ (Su) \int_X f \ d\mu = \sup_{\alpha > 0} M \{\alpha, \mu(f \geq \alpha)\} \]
Level sets and level measure

\[ \{ x \in X ; \ f(x) \geq \alpha \} \]

\[ h_{\mu,f}(\alpha) = \mu(f \geq \alpha) := \mu(\{ x \in X ; \ f(x) \geq \alpha \}) \]

Integrals

- The Lebesgue integral (L) \[ \int_X f \, d\mu = \int_0^\infty \mu(f \geq \alpha) \, d\alpha \]
- The Choquet integral (Ch) \[ \int_X f \, d\mu = \int_0^\infty \mu(f \geq \alpha) \, d\alpha \]
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Integrals

- **The Lebesgue integral**  \( (L) \int_X f \, d\mu = \int_0^\infty \mu(f \geq \alpha) \, d\alpha \)

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- **The Sugeno integral**  \( (Su) \int_X f \, d\mu = \sup_{\alpha > 0} M \{ \alpha, \mu(f \geq \alpha) \} \)
(Ch) \[ \int_X h \, d\mu := \int_0^\infty \mu(\{x \in X : h(x) > \alpha\}) \, d\alpha \]

(Ch) \[ \int x_i \, d\mu = x_m \cdot \mu\{m, p, e\} + (x_e - x_m) \cdot \mu\{p, e\} + (x_p - x_e) \cdot \mu\{p\} \]

\[ = x_m \cdot (\mu\{m, p, e\} - \mu\{p, e\}) + x_e \cdot (\mu\{p, e\} - \mu\{p\}) + x_p \cdot \mu\{p\} \]
(Ch) \[
\int_X h \, d\mu := \int_0^\infty \mu \left( \{ x \in X : h(x) > \alpha \} \right) \, d\alpha
\]

(Ch) \[
\int_{\{m,p,e\}} x_i \, d\mu = x_m \cdot \mu \{m,p,e\} + (x_e - x_m) \cdot \mu \{p,e\} + (x_p - x_e) \cdot \mu \{p\}
\]

\[
= x_m \cdot \left( \mu \{m,p,e\} - \mu \{p,e\} \right) + x_e \cdot \left( \mu \{p,e\} - \mu \{p\} \right) + x_p \cdot \mu \{p\}
\]

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\[
\int_{\{m,p,e\}} x_i \, d\# = \frac{1}{3} \cdot 11 + \frac{1}{3} \cdot 15 + \frac{1}{3} \cdot 16
\]
\[
\int_{\mathcal{X}} x_i \, d\mu = 0.2 \cdot 19 + 0.4 \cdot 12 + 0.4 \cdot 11
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\[
\nu\{m\} = 0,4 \\
\nu\{p\} = 0,4 \\
\nu\{e\} = 0,2 \\
\nu\{m, p\} = 0,6 \\
\nu\{m, e\} = 0,9 \\
\nu\{p, e\} = 0,9 \\
\nu\{m, p, e\} = 1
\]
A new concept - three steps to super level measures

| metric space $X$ | premeasure $\sigma : \mathcal{E} \subseteq \mathcal{E}_B \rightarrow [0, \infty)$ |

(A) **Size.** a function $s : \mathcal{B}(X) \rightarrow [0, +\infty]^{\mathcal{E}}$ satisfying

(i) if $|f| \leq |g|$, then $s(f)(a) \leq s(g)(a)$;

(ii) $s(\lambda f)(a) = |\lambda| s(f)(a)$ for each $\lambda \in \mathbb{C}$;

(iii) $s(f + g)(a) \leq C_s s(f)(a) + C_s s(g)(a)$ for some fixed $C_s \geq 1$ depending only on $s$.

A triple $(X, \sigma, s)$ is called an outer measure space.

(B) **Outer essential supremum.** $f \in \mathcal{B}(X), b \in \mathcal{E}_B$

$$\text{outsup}_b s(f) := \sup \{s(f1_b)(a); a \in \mathcal{E}\}$$

(C) **Super level measure.** $(X, \sigma, s), f \in \mathcal{B}(X), \alpha > 0$

$$\mu(s(f) > \alpha) := \inf \left\{ \mu(b) : b \in \mathcal{E}_B, \text{outsup}_{X \setminus b} s(f) \leq \alpha \right\}$$
Is a new concept useful?


- natural $L^p$ theory for outer measures offers a unifying language for both Carleson measure and time-frequency analysis
- gaining a streamlined view on time-frequency analysis was the original motivation for their paper
- the outcome of a long evolution process
- a point of the paper is that in many examples of their interest the bound is a Hölder inequality with respect to an outer measure

Is a new concept useful outside of functional analysis?

Hopefully …
Three steps to super level measures - Y. Do and C. Thiele, modified

(A) **Size.** a function $s : \mathcal{B}(X) \to [0, \infty]$ satisfying

(i) if $|f| \leq |g|$, then $s(f)(a) \leq s(g)(a)$;
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A triple $(X, \sigma, s)$ is called an outer measure space.

(B) **Outer essential supremum.** $f \in \mathcal{B}(X), \ F \in \mathcal{E}_B$

$$\text{outsup}_b s(f)(E) := \sup \{ s(f 1_b)(a) : a \in E \}$$

(C) **Super level measure.** $(X, \sigma, s), \ f \in \mathcal{B}(X), \ \alpha > 0$ \quad $(X, \mathcal{E}, s)$

$$\mu(s(f)(E) > \alpha) := \inf \left\{ \mu(b) : b \in \mathcal{E}_B, \ \text{outsup}_{X \setminus b} s(f) \leq \alpha \right\}$$
An example

\[ \text{An example} \]

\[ \text{s}_{\text{int}}(f)(a) = (L) \int_a |f| \, d\mu \]

(A) **Size.** a function \( s : \mathcal{B}(X) \to [0, +\infty]^{E_B} \) satisfying

(i) if \( |f| \leq |g| \), then \( s(f)(a) \leq s(g)(a) \);

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(iii) \( s(f + g)(a) \leq C_s s(f)(a) + C_s s(g)(a) \) for some fixed \( C_s \geq 1 \) depending only on \( s \).
(B) **Outer essential supremum.** \( f \in \mathcal{B}(X), b \in E_B, \text{ reasonable } E \)

\[
\text{outsup}_{b} s_{\text{int}}(f)\langle E \rangle := \sup\{s_{\text{int}}(f1_{b})(a) : a \in E\}
\]

\[
= \sup\{(L) \int_{a \cap b} f \, d\mu : a \in E\}
\]

\[
= (L) \int_{b} f \, d\mu = s_{\text{int}}(f)(b)
\]
An example

\[
\begin{align*}
\text{s}_{\text{int}}( f )(a) &= (L) \int_{a} |f| \, d\mu \\
\end{align*}
\]

(B) **Outer essential supremum.** \( f \in \mathcal{B}(X), \ b \in E_B, \) reasonable \( E \)

\[
\text{outsup}_{b} \text{ s}_{\text{int}}( f ) \langle E \rangle := \sup \{ \text{s}_{\text{int}}( f 1_b)(a) : a \in E \} \\
= \sup \{ (L) \int_{a \cap b} f \, d\mu : a \in E \} \\
= (L) \int_{b} f \, d\mu = \text{s}_{\text{int}}( f )(b)
\]
An example

\[ s_{\text{int}}(f)(a) = (L) \int_a |f| \, d\mu \]

(B) **Outer essential supremum.** \( f \in \mathcal{B}(X), \ b \in \mathbf{E}_B, \) reasonable \( \mathbf{E} \)

\[
\text{outsup}_b s_{\text{int}}(f)(\mathbf{E}) := \sup \{ s_{\text{int}}(f1_a)(a) : a \in \mathbf{E} \}
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An example

\[ s_{\text{int}}(f)(a) = (L) \int_a^b |f| \, d\mu \]

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\text{s_{int}}(f)(a) = (L) \int_{a} |f| \, d\mu
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An example

\[ s_{\text{int}}(f)(a) = (L) \int_a |f| \, d\mu \]

(C) Super level measure. \((X, E, s), f \in \mathcal{B}(X), \alpha > 0\)

\[ \mu(s_{\text{int}}(f)(E) > \alpha) := \inf \left\{ \mu(b) : b \in E_B, \ \overset{\text{outsup}_{X \setminus b}}{\text{sup}} s_{\text{int}}(f) \leq \alpha \right\} \]

\[ = \inf \left\{ \mu(b) : b \in E_B, \ (L) \int_{X \setminus b} f \, d\mu \leq \alpha \right\} \]
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s_{\text{int}}(f)(a) = (L) \int_a |f| \, d\mu
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An example

\[ \text{s}_{\text{int}}(f)(a) = (L) \int_{a}^{\infty} |f| \, d\mu \]

(C) Super level measure. \( (X, E, s), f \in \mathcal{B}(X), \alpha > 0 \)

\[ \mu(\text{s}_{\text{int}}(f) \langle E \rangle > \alpha) := \inf \left\{ \mu(b) : b \in E_B, \text{outsup}_{X \setminus b} \text{s}_{\text{int}}(f) \leq \alpha \right\} \]

\[ = \inf \left\{ \mu(b) : b \in E_B, (L) \int_{X \setminus b} f \, d\mu \leq \alpha \right\} \]
Is the new concept a generalization of the original one?

\[ s_\infty(f)(a) = \sup |f|[a] \]

**(A) Size.** a function \( s : \mathcal{B}(X) \to [0, +\infty]^E_B \) satisfying

(i) if \(|f| \leq |g|\), then \( s(f)(a) \leq s(g)(a) \);

(ii) \( s(\lambda f)(a) = |\lambda| s(f)(a) \) for each \( \lambda \in \mathbb{C} \);

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\[ s_\infty(f)(a) = \sup |f|[a] \]

(B) **Outer essential supremum.** \( f \in \mathcal{B}(X) \), \( b \in \mathcal{E}_B \), reasonable \( \mathcal{E} \)

\[
\text{outsup}_{b} s_\infty(f)(E) := \sup \{ s_\infty(f1_b)(a) : a \in \mathcal{E} \}
\]

\[ = \sup \{ \sup |f|[a \cap b] : a \in \mathcal{E} \} \]

\[ = \sup |f|[b] = s_\infty(f)(b) \]
Is the new concept a generalization of the original one?

\[ s_\infty(f)(a) = \sup |f|[a] \]

(B) Outer essential supremum. \( f \in \mathcal{B}(X), \ b \in E_B, \) reasonable \( E \)

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Is the new concept a generalization of the original one?

\[ s_\infty(f)(a) = \sup |f|_a \]

(B) **Outer essential supremum.** \( f \in \mathcal{B}(X), \; b \in E_B \), reasonable \( E \)

\[
\text{outsup}_{b} s_\infty(f) \langle E \rangle := \sup \{ s_\infty(f \mathbf{1}_b)(a) : \; a \in E \} \\
= \sup \{ \sup |f|[a \cap b] : \; a \in E \} \\
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\[ s_\infty(f)(a) = \sup |f|[a] \]

\( \text{(C) Super level measure.} \quad (X, E, s), f \in \mathcal{B}(X), \alpha > 0 \)

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\mu(s_{\text{int}}(f)(E) > \alpha) := \inf \left\{ \mu(b) : b \in E_B, \ \text{outrsup} \ s_\infty(f) \leq \alpha \right\}
\]

\[
= \inf \{ \mu(b) : b \in E_B, \ \sup |f|[b] \leq \alpha \} \]
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\[ s_\infty(f)(a) = \sup |f|[a] \]

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\[ \mu(\text{sint}(f)(E) > \alpha) := \inf \left\{ \mu(b) : b \in E_B, \sup_{X \setminus b} \text{outsup} s_\infty(f) \leq \alpha \right\} \]

\[ = \inf \{ \mu(b) : b \in E_B, \sup |f|[b] \leq \alpha \} \]
Is the new concept a generalization of the original one?

For each $\alpha > 0$ we may write

$$h_{\mu,f}(\alpha) = \mu(\{x \in X; |f(x)| > \alpha\}) = \inf \left\{ \mu(b) : b \in \mathbf{E}_B, \left(\forall x \in X \setminus b\right) |f(x)| \leq \alpha \right\}.$$

$X$

$\{x \in X; f(x) > \alpha\}$

$\{x \in X; f(x) \leq \alpha\}$
Is the new concept a generalization of the original one?

For each $\alpha > 0$ we may write

$$h_{\mu,f}(\alpha) = \mu\left(\{x \in X; |f(x)| > \alpha\}\right) = \inf \left\{ \mu(b): b \in E_B, \sup_{x \in X \setminus b} |f(x)| \leq \alpha \right\}.$$
How far is the new concept from the old one?

$$s_{sum}(f)(a) = \sum_{i \in a} |f(i)|$$

$$\mu(s_{sum}(f)(E) > \alpha) = \inf \left\{ \mu(b) : \sum_{i \in X \setminus a} |f(i)| \leq \alpha \right\}$$

$$\mu(s(f)(E) > \alpha) = m(\{x \in X; |G_f(x)| > \beta \alpha\})?$$

$$X = \{a, b, c\}$$

We assume that $\mu$ is strictly increasing with respect to the following order $\prec$ on $E_{power}$:

$$\emptyset \prec \{a\} \prec \{b\} \prec \{c\} \prec \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.$$  

We define a function $f$ on $X$ as $f(a) = 2$, $f(b) = 3$, $f(c) = 4$. 

$$X = \{a, b, c\}$$
How far is the new concept from the old one?

\[ \text{s}_{\text{sum}}(f)(a) = \sum_{i \in a} |f(i)| \]

\[ \mu(\text{s}_{\text{sum}}(f) \langle E \rangle > \alpha) = \inf \left\{ \mu(b) : \sum_{i \in X \setminus a} |f(i)| \leq \alpha \right\} \]

?\[ \mu(s(f) \langle E \rangle > \alpha) = m(\{x \in X; |G_f(x)| > \beta_\alpha\})? \]

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We assume that \( \mu \) is strictly increasing with respect to the following order \( \prec \) on \( E_{\text{power}} \):

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\[ ?\mu(s(f)(E) > \alpha) = m(\{x \in X; |G_f(x)| > \beta \alpha\})? \]

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\text{s}_{\text{sum}}(f)(a) = \sum_{i \in a} |f(i)|
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?\mu(\text{s}(f)(E) > \alpha) = m(\{x \in X; |G_f(x)| > \beta \alpha\})?
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X = \{a, b, c\}
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We assume that \(\mu\) is strictly increasing with respect to the following order \(\prec\) on \(E_{\text{power}}\): \[
\emptyset \prec \{a\} \prec \{b\} \prec \{c\} \prec \{a, b\} \prec \{a, c\} \prec \{b, c\} \prec X.
\]

We define a function \(f\) on \(X\) as \(f(a) = 2, f(b) = 3, f(c) = 4.\)
How far is the new concept from the old one?

Topological space $X$, Borel subsets $E_B$, monotone measure $\mu : E_B \to [0, +\infty]$

- A new underlying set $E_B$.
- A new induced monotone measure $m_\mu : 2^{E_B} \to [0, +\infty]$

$$m_\mu(F) := \inf\{\mu(a) : a \in E_B \setminus F\}.$$

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**Proposition**

Let $(X, E, s)$ be a sub-Borel size space. Then for every $f \in \mathcal{B}(X)$ we have

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**Proof.** Just different notation:

$$\mu(s(f)(E) > \alpha) = \inf \left\{ \mu(a) : a \in E_B, \overset{\text{outsup}}{X \setminus a} s(f)(E) \leq \alpha \right\} =$$

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How far is the new concept from the old one?

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\]

Proposition

Let \((X, \mathcal{E}, s)\) be a sub-Borel size space. Then for every \( f \in \mathcal{B}(X) \) we have

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How good is the induced measure?

\[ m_\mu(F) := \inf \{ \mu(a) : a \in E_B \setminus F \}. \]

**Proposition**

Let \( X \) be a topological space and \( \mu \) be a monotone measure on \( E_B \).

(a) \( m_\mu \) is superadditive.

(b) \( m_\mu(\bigcap A) = \inf \{ m_\mu(A) : A \in \mathcal{A} \} \) for any \( \mathcal{A} \subseteq 2^{E_B} \).

(c) \( m_\mu \) is upper semicontinuous.
How good is the induced measure?

\[ m_\mu(F) := \inf\{\mu(a) : a \in E_B \setminus F\}. \]

**Lemma**

Let \( X \) be a topological space and \( \mu \) be a monotone measure on \( E_B \).

(a) If \( \mu(a) = 0 \) and \( a \not\in F \) then \( m_\mu(F) = 0 \).

(b) If \( \emptyset \not\in F \) then \( m_\mu(F) = 0 \).

(c) If \( a \neq \emptyset \) then \( m_\mu(\{a\}) = 0 \).

(d) If \( |N_\mu| > 1 \) then \( m_\mu(\emptyset) = 0 \).

(e) \( m_\mu(F) = 0 \) if and only if for any \( \varepsilon > 0 \) there is \( a \not\in F \) such that \( \mu(a) < \varepsilon \).

(f) \( m_\mu(E_B) = \mu(X) \).

(g) \( \mu(a) = m_\mu(E_B \setminus \{a\}) \).

(h) \( m_\mu(F) = \inf\{m_\mu(E_B \setminus \{a\}) : F \subseteq E_B \setminus \{a\}\} \).
What are properties of the induced function?

\[ t_f(a) := \operatorname{outsup}_{X \setminus a} s(f)(E). \]

Proposition

Let \((X, \mathcal{E}, s)\) be a sub-Borel size space, then for every \(f \in \mathcal{B}(X)\) we have

(a) \(t_f(\emptyset) = \sup_{E \in \mathcal{E}} s(f)(E) \) and \(t_f(X) = 0\).

(b) \(t_f\) is anti-monotone, i.e., \(t_f(a_2) \leq t_f(a_1)\) whenever \(a_1 \subseteq a_2\).

(c) If \(a_1, a_2 \in \mathcal{E}_B\) then \(t_f(a_1 \cap a_2) \leq C_s(t_f(a_1) + t_f(a_2))\).
What is next?

Question
What are properties of the smallest \( \sigma \)-algebra on \( E_B \) such that all \( t_f \) are measurable?

Question
Which topologies on \( E_B \) do make \( t_f \) Borel on \( E_B \)?

Question
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Thanks for Your attention!