A long chain of P-points

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P-points

Definition

Ultrafilter $\mathcal{U}$ on $\omega$ is a **P-point** if and only if for every countable collection $\{a_n : n < \omega\} \subset \mathcal{U}$ there is an $a \in \mathcal{U}$ such that $a \subset^* a_n$ for each $n < \omega$.

In the fifties, W. Rudin showed that they consistently exist; in the seventies, Shelah showed that their existence cannot be proved in ZFC only; Choquet calls them ultrafiltres absolument 1-simples or $\delta$-stables; MA($\sigma$-centered) implies the existence of $2^c$ P-points (both CH and PFA imply MA($\sigma$-centered)); there are equivalent definitions which rely on model theoretic and topological notions.
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- Choquet calls them ultrafiltres absolument 1-simples or $\delta$-stables;
- MA($\sigma$-centered) implies the existence of $2^c$ P-points (both CH and PFA imply MA($\sigma$-centered));
- there are equivalent definitions which rely on model theoretic and topological notions.
Rudin-Keisler ordering

Definition

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on $\omega$. We say that $\mathcal{U}$ is **Rudin-Keisler reducible** to $\mathcal{V}$ if there is a map $f : \omega \to \omega$ such that for every $a \subset \omega$, $a \in \mathcal{U} \iff f^{-1}[a] \in \mathcal{V}$. 

- If $\mathcal{U}$ is Rudin-Keisler reducible to $\mathcal{V}$ we write $\mathcal{U} \leq_{RK} \mathcal{V}$;
- Relation $\leq_{RK}$ is a quasi-ordering on the set of all ultrafilters on $\omega$;
- If both $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$ hold, we say that $\mathcal{U}$ and $\mathcal{V}$ are **Rudin-Keisler equivalent** ($\mathcal{U} \equiv_{RK} \mathcal{V}$).
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Rudin-Keisler ordering

Theorem (Kunen 1972)
There are ultrafilters \( U \) and \( V \) on \( \omega \) such that \( U \not\leq_{RK} V \) and \( V \not\leq_{RK} U \).

Theorem (Keisler)
CH implies the existence of \( 2^c \) RK-incomparable selective ultrafilters.

Theorem (Blass 1973)
MA(\( \sigma \)-centered) implies that \( \omega_1 \) embeds into \( \langle R, \leq_{RK} \rangle \).

Problem (Blass 1973)
Is there a \( c^+ \)-chain of P-points under Rudin-Keisler ordering?

Theorem (Rosen 1985)
CH implies that \( \omega_1 \) embeds as an initial segment into \( \langle R, \leq_{RK} \rangle \).
Tukey ordering

Definition

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on $\omega$. We say that $\mathcal{U}$ is Tukey reducible to $\mathcal{V}$ if there is a monotone map $\phi : \mathcal{V} \to \mathcal{U}$ which is cofinal in $\mathcal{U}$. 

- If $\mathcal{U}$ is Tukey reducible to $\mathcal{V}$, we write $\mathcal{U} \leq_T \mathcal{V}$.
- If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$, then we say that $\mathcal{U}$ and $\mathcal{V}$ are Tukey equivalent ($\mathcal{U} \equiv_T \mathcal{V}$).
- If $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U} \leq_T \mathcal{V}$.
- $\mathcal{C}$ has maximal Tukey type among all directed sets of cardinality $\mathfrak{c}$. 

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Tukey ordering

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- If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$, then we say that $\mathcal{U}$ and $\mathcal{V}$ are Tukey equivalent ($\mathcal{U} \equiv_T \mathcal{V}$);
- If $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U} \leq_T \mathcal{V}$;
- $[\mathfrak{c}]^{<\omega}$ has maximal Tukey type among all directed sets of cardinality $\mathfrak{c}$. 
Tukey ordering

Theorem (Isbell 1965)

There is an ultrafilter $\mathcal{U}$ on $\omega$ such that $\mathcal{U} \equiv_T [c]^{<\omega}$. 

Problem (Isbell 1965)

Is there an ultrafilter which is not Tukey equivalent to $[c]^{<\omega}$?
Tukey ordering

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**Tukey ordering**

**Theorem (Dobrinen-Todorčević 2011)**

Suppose that \( \mathcal{V} \) is a P-point, \( \mathcal{U} \) an arbitrary ultrafilter on \( \omega \), and \( \mathcal{U} \leq_T \mathcal{V} \). Then there is a continuous monotone map \( \phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega) \) such that \( \phi \upharpoonright \mathcal{V} \) is cofinal in \( \mathcal{U} \).

This implies that every P-point can have at most \( c \) predecessors.

**Theorem (Dobrinen-Todorčević 2011)**

CH implies that \( c \) embeds into \( \langle R, \leq_T \rangle \).

**Problem (Dobrinen-Todorčević 2011)**

Is there a \( c^+ \)-sequence of P-points under Tukey ordering?
Tukey ordering

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- This implies that every P-point can have at most $\mathfrak{c}$ predecessors.
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Embedding of $\mathcal{P}(\omega)/\text{fin}$

**Theorem (Raghavan-Shelah 2017)**

$\text{MA}(\sigma\text{-centered})$ implies that $\langle \mathcal{P}(\omega)/\text{Fin}, \subset^* \rangle$ embeds into both $\langle \mathcal{R}, \leq_T \rangle$ and $\langle \mathcal{R}, \leq_{RK} \rangle$.

- In particular this implies that any poset of size $\mathfrak{c}$ can be embedded into $\langle \mathcal{R}, \leq_{RK} \rangle$ and $\langle \mathcal{R}, \leq_T \rangle$. 
Theorem (K-Raghavan)

CH implies that there is a $c^+$ sequence of P-points under both Rudin-Keisler and Tukey ordering.
Theorem (K-Raghavan)

CH implies that there is a $c^+$ sequence of P-points under both Rudin-Keisler and Tukey ordering.

- Key notion: $\delta$-generic sequence of P-points;
- We only have to worry about $c$ many Tukey maps;
- In a forthcoming paper of D. Raghavan and J. Verner, a simpler proof of this result will be presented.
Basic poset

Definition

$\mathbb{P}$ is the set of all sequences $c : \omega \rightarrow [\omega]^\omega \setminus \{0\}$ such that for each $n \in \omega$, both $|c(n)| < |c(n + 1)|$ and $\max(c(n)) < \min(c(n + 1))$ hold.

If $c, d \in \mathbb{P}$, then $c \leq d$ if there is an $l < \omega$ such that

$$\forall m \geq l \exists n \geq m [c(m) \subset d(n)].$$

\footnote{set$(c)$ denotes the set $\bigcup_{n<\omega} c(n)$}
Basic poset

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\]

- Note that if \( c \leq d \), then \( \text{set}(c) \subset^* \text{set}(d) \).
Adding an ultrafilter on top

Definition

Let $Q^\delta$ be the set of all $q = \langle c_q, \gamma_q, X_q, \langle \pi_{q,\alpha} : \alpha \in X_q \rangle \rangle$ such that:

- $c_q \in \mathcal{P}$;
- $\gamma_q \leq \delta$;
- $X_q \in [\delta]^{\leq \omega}$ is such that $\gamma_q = \sup(X_q)$, and that $\gamma_q \in X_q$ iff $\gamma_q < \delta$;
- $\pi_{q,\alpha}$ ($\alpha \in X_q$) are maps from $\omega^\omega$ such that:
  - $\pi''_{q,\alpha} \text{ set}(c_q) \in U_\alpha$;
  - $\forall \alpha, \beta \in X_q \ [\alpha \leq \beta \Rightarrow \forall \omega k \in \text{ set}(c_q) \ [\pi_{q,\alpha}(k) = \pi_{\beta,\alpha}(\pi_{q,\beta}(k))]]$;
  - there are $\psi_{q,\alpha} \in \omega^\omega$ and $b_{q,\alpha} \geq c_q$ such that $\langle \pi_{q,\alpha}, \psi_{q,\alpha}, b_{q,\alpha} \rangle$ is a normal triple;

Ordering on $Q^\delta$ is given by: $q_1 \leq q_0$ if and only if

- $c_{q_1} \leq c_{q_0}$, and $X_{q_1} \supset X_{q_0}$, and for every $\alpha \in X_{q_0}$, $\pi_{q_1,\alpha} = \pi_{q_0,\alpha}$.
\(\delta\)-generic sequence of P-points

Let \(\delta \leq \omega_2\). We call a sequence \(\langle \langle c^\alpha_i : i < c, \alpha < \delta \rangle, \langle \pi_{\beta,\alpha} : \alpha \leq \beta < \delta \rangle \rangle\) \(\delta\)-generic if and only if:

- for every \(\alpha < \delta\), \(\langle c^\alpha_i : i < c \rangle\) is a decreasing sequence in \(\mathbb{P}\); for every \(\alpha \leq \beta < \delta\), \(\pi_{\beta,\alpha} \in \omega^\omega\);

- for every \(\alpha < \delta\), \(U_\alpha = \{ a \in \mathcal{P}(\omega) : \exists i < c \ [\text{set}(c^\alpha_i) \subset^* a] \}\) is an ultrafilter on \(\omega\) and it is a rapid\(^2\) P-point;

- for every \(\alpha < \beta < \delta\), every normal triple \(\langle \pi_1, \psi_1, b_1 \rangle\) and every \(d \leq b_1\) if \(\pi_1'' \text{set}(d) \in U_\alpha\), then for every \(a \in U_\beta\) there is \(b \in U_\beta\) such that \(b \subset^* a\) and that there are \(\pi, \psi \in \omega^\omega\) and \(d^* \leq_0 d\) so that \(\langle \pi, \psi, d^* \rangle\) is a normal triple, \(\pi'' \text{set}(d^*) = b\) and \(\forall k \in \text{set}(d^*) \ [\pi_1(k) = \pi_{\beta,\alpha}(\pi(k))]\).

\(^2\mathcal{U}\) is rapid if for every \(f \in \omega^\omega\) there is \(X \in \mathcal{U}\) such that \(X(n) \geq f(n)\) for each \(n < \omega\)
\(\delta\)-generic sequence of P-points

- if \(\alpha < \beta < \delta\), then \(U_\beta \not\leq_T U_\alpha\).
- for every \(\alpha < \delta\), \(\pi_{\alpha,\alpha} = \text{id}\) and:
  - \(\forall \alpha \leq \beta < \delta \ \forall i < c \ [\pi''_{\beta,\alpha} \ \text{set}(c^\beta_i) \in U_\alpha]\);
  - \(\forall \alpha \leq \beta \leq \gamma < \delta \ \exists i < c \ \forall \kappa \in \text{set}(c^\gamma_i) \ [\pi_{\gamma,\alpha}(\kappa) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\kappa))];\)
  - for \(\alpha < \beta < \delta\) there are \(i < c\), \(b_{\beta,\alpha} \in \mathbb{P}\) and \(\psi_{\beta,\alpha} \in \omega^\omega\) such that \(\langle \pi_{\beta,\alpha}, \psi_{\beta,\alpha}, b_{\beta,\alpha} \rangle\) is a normal triple and \(c^\beta_i \leq b_{\beta,\alpha}\).
δ-generic sequence of P-points

• if \( \mu < \delta \) is a limit ordinal such that \( \text{cof}(\mu) = \omega \), \( X \subset \mu \) is a countable set such that \( \text{sup}(X) = \mu \), \( \langle d_j : j < \omega \rangle \) is a decreasing sequence of conditions in \( \mathbb{P} \), and \( \langle \pi_\alpha : \alpha \in X \rangle \) is a sequence of maps in \( \omega^\omega \) such that:
  
  ▶ \( \forall \alpha \in X \ \forall j < \omega \ [\pi''_\alpha \text{ set}(d_j) \in \mathcal{U}_\alpha] \);  
  
  ▶ \( \forall \alpha, \beta \in X \ [\alpha \leq \beta \Rightarrow \exists j < \omega \ \forall^\infty k \in \text{set}(d_j) \ [\pi_\alpha(k) = \pi_{\beta,\alpha}(\pi_{\beta}(k))] \]};  
  
  ▶ for all \( \alpha \in X \) there are \( j < \omega \), \( b_\alpha \in \mathbb{P} \) and \( \psi_\alpha \in \omega^\omega \) such that \( \langle \pi_\alpha, \psi_\alpha, b_\alpha \rangle \) is a normal triple and \( d_j \leq b_\alpha \); 

then the set of all \( i^* < c \) such that there are \( d^* \in \mathbb{P} \) and \( \pi, \psi \in \omega^\omega \) satisfying:
  
  ▶ \( \forall j < \omega \ [d^* \leq d_j] \) and \( \text{set}(c_{i^*}^\mu) = \pi'' \text{ set}(d^*) \);  
  
  ▶ \( \forall \alpha \in X \ \forall^\infty k \in \text{set}(d^*) \ [\pi_\alpha(k) = \pi_{\mu,\alpha}(\pi(k))] \]};  
  
  ▶ \( \langle \pi, \psi, d^* \rangle \) is a normal triple;

is cofinal in \( c \);
Problem

Which partial orders can be embedded into $\langle \mathcal{R}, \leq_{RK} \rangle$ under suitable hypothesis?

• Such a poset must be of cardinality at most $2^c$;
• Such a poset must be locally $c$, i.e. each element can have only $c$ many predecessors.

It is plausible that these are the only two obstacles.
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- It is plausible that these are the only two obstacles.