A long chain of P-points

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P-points

Definition

Ultrafilter $\mathcal{U}$ on ω is a *P-point* if and only if for every countable collection $\{a_n : n < \omega\} \subset \mathcal{U}$ there is an $a \in \mathcal{U}$ such that $a \subset^* a_n$ for each $n < \omega$.

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- In the fifties, W. Rudin showed that they consistently exist;
- In the seventies, Shelah showed that their existence cannot be proved in ZFC only;
- Choquet calls them *ultrafiltres absolument 1-simples* or *δ-stables*;
- MA(σ-centered) implies the existence of $2^{\mathfrak{c}}$ P-points (both CH and PFA imply MA(σ-centered));
- there are equivalent definitions which rely on model theoretic and topological notions.

Rudin-Keisler ordering

Definition

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω. We say that $\mathcal{U}$ is *Rudin-Keisler reducible* to $\mathcal{V}$ if there is a map $f : \omega \to \omega$ such that for every $a \subset \omega$, $a \in \mathcal{U} \Leftrightarrow f^{-1}[a] \in \mathcal{V}$.

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- If $\mathcal{U}$ is Rudin-Keisler reducible to $\mathcal{V}$ we write $\mathcal{U} \leq_{RK} \mathcal{V}$;
- Relation $\leq_{RK}$ is a quasi-ordering on the set of all ultrafilters on ω;
- If both $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$ hold, we say that $\mathcal{U}$ and $\mathcal{V}$ are Rudin-Keisler equivalent ($\mathcal{U} \equiv_{RK} \mathcal{V}$);

Rudin-Keisler ordering

Theorem (Kunen 1972)

There are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω such that $\mathcal{U} \not\leq_{RK} \mathcal{V}$ and $\mathcal{V} \not\leq_{RK} \mathcal{U}$.

Theorem (Keisler)

CH implies the existence of $2^{\mathfrak{c}}$ RK-incomparable selective ultrafilters.

Theorem (Blass 1973)

MA(σ-centered) implies that ω_1 embeds into $\langle \mathcal{R}, \leq_{RK} \rangle$.

Problem (Blass 1973)

Is there a $\mathfrak{c}^+$-chain of P-points under Rudin-Keisler ordering?

Theorem (Rosen 1985)

CH implies that ω_1 embeds as an initial segment into $\langle \mathcal{R}, \leq_{RK} \rangle$.

Tukey ordering

Definition

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω. We say that $\mathcal{U}$ is *Tukey reducible* to $\mathcal{V}$ if there is a monotone map $\phi : \mathcal{V} \to \mathcal{U}$ which is cofinal in $\mathcal{U}$.

Tukey ordering

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Suppose that $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω. We say that $\mathcal{U}$ is *Tukey reducible* to $\mathcal{V}$ if there is a monotone map $\phi : \mathcal{V} \to \mathcal{U}$ which is cofinal in $\mathcal{U}$.

- If $\mathcal{U}$ is Tukey reducible to $\mathcal{V}$, we write $\mathcal{U} \leq_T \mathcal{V}$;
- If $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$, then we say that $\mathcal{U}$ and $\mathcal{V}$ are Tukey equivalent ($\mathcal{U} \equiv_T \mathcal{V}$);
- If $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U} \leq_T \mathcal{V}$;
- $[\mathfrak{c}]^{<\omega}$ has maximal Tukey type among all directed sets of cardinality $\mathfrak{c}$.

Tukey ordering

Theorem (Isbell 1965)

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Tukey ordering

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Problem (Isbell 1965)

Is there an ultrafilter which is not Tukey equivalent to $[\mathfrak{c}]^{<\omega}$?

Tukey ordering

Theorem (Dobrinen-Todorčević 2011)

Suppose that $\mathcal{V}$ is a P-point, $\mathcal{U}$ an arbitrary ultrafilter on ω, and $\mathcal{U} \leq_T \mathcal{V}$. Then there is a continuous monotone map $\phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\phi \restriction \mathcal{V}$ is cofinal in $\mathcal{U}$.

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- This implies that every P-point can have at most $\mathfrak{c}$ predecessors.

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Problem (Dobrinen-Todorčević 2011)

Is there a $\mathfrak{c}^+$-sequence of P-points under Tukey ordering?

Embedding of $\mathcal{P}(\omega)/fin$

Theorem (Raghavan-Shelah 2017)

MA(σ-centered) implies that $\langle \mathcal{P}(\omega)/Fin, \subset^* \rangle$ embeds into both $\langle \mathcal{R}, \leq_T \rangle$ and $\langle \mathcal{R}, \leq_{RK} \rangle$.

- In particular this implies that any poset of size $\mathfrak{c}$ can be embedded into $\langle \mathcal{R}, \leq_{RK} \rangle$ and $\langle \mathcal{R}, \leq_T \rangle$.

A long chain

Theorem (K-Raghavan)

CH implies that there is a $\mathfrak{c}^+$ sequence of P-points under both Rudin-Keisler and Tukey ordering.

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- Key notion: δ-generic sequence of P-points;
- We only have to worry about $\mathfrak{c}$ many Tukey maps;
- In a forthcoming paper of D. Raghavan and J. Verner, a simpler proof of this result will be presented.

Basic poset

Definition

$\mathbb{P}$ is the set of all sequences $c : \omega \to [\omega]^{<\omega} \setminus \{0\}$ such that for each $n \in \omega$, both $|c(n)| < |c(n+1)|$ and $\max(c(n)) < \min(c(n+1))$ hold.

If $c, d \in \mathbb{P}$, then $c \leq d$ if there is an $l < \omega$ such that

$$\forall m \geq l \; \exists n \geq m \; [c(m) \subset d(n)].$$

[1] set(c) denotes the set $\bigcup_{n<\omega} c(n)$

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- Note that if $c \leq d$, then $\mathrm{set}(c) \subset^* \mathrm{set}(d)$[1].

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Adding an ultrafilter on top

Definition

Let $\mathbb{Q}^\delta$ be the set of all $q = \langle c_q, \gamma_q, X_q, \langle \pi_{q,\alpha} : \alpha \in X_q \rangle \rangle$ such that:

- $c_q \in \mathbb{P}$;
- $\gamma_q \leq \delta$;
- $X_q \in [\delta]^{\leq\omega}$ is such that $\gamma_q = \sup(X_q)$, and that $\gamma_q \in X_q$ iff $\gamma_q < \delta$;
- $\pi_{q,\alpha}$ $(\alpha \in X_q)$ are maps from ω^ω such that:
  - $\pi''_{q,\alpha} set(c_q) \in \mathcal{U}_\alpha$;
  - $\forall \alpha, \beta \in X_q \;\; [\alpha \leq \beta \Rightarrow \forall^\infty k \in set(c_q) \; [\pi_{q,\alpha}(k) = \pi_{\beta,\alpha}(\pi_{q,\beta}(k))]]$;
  - there are $\psi_{q,\alpha} \in \omega^\omega$ and $b_{q,\alpha} \geq c_q$ such that $\langle \pi_{q,\alpha}, \psi_{q,\alpha}, b_{q,\alpha} \rangle$ is a normal triple;

Ordering on $\mathbb{Q}^\delta$ is given by: $q_1 \leq q_0$ if and only if

$$c_{q_1} \leq c_{q_0}, \text{ and } X_{q_1} \supset X_{q_0}, \text{ and for every } \alpha \in X_{q_0}, \; \pi_{q_1,\alpha} = \pi_{q_0,\alpha}.$$

δ-generic sequence of P-points

Let $\delta \leq \omega_2$. We call a sequence $\langle\langle c_i^\alpha : i < \mathfrak{c}, \alpha < \delta\rangle, \langle\pi_{\beta,\alpha} : \alpha \leq \beta < \delta\rangle\rangle$ *δ-generic* if and ony if:

- for every $\alpha < \delta$, $\langle c_i^\alpha : i < \mathfrak{c}\rangle$ is a decreasing sequence in $\mathbb{P}$; for every $\alpha \leq \beta < \delta$, $\pi_{\beta,\alpha} \in \omega^\omega$;
- for every $\alpha < \delta$, $\mathcal{U}_\alpha = \{a \in \mathcal{P}(\omega) : \exists i < \mathfrak{c}\ [\text{set}(c_i^\alpha) \subset^* a]\}$ is an ultrafilter on ω and it is a rapid[2] P-point;
- for every $\alpha < \beta < \delta$, every normal triple $\langle\pi_1, \psi_1, b_1\rangle$ and every $d \leq b_1$ if $\pi_1'' \text{set}(d) \in \mathcal{U}_\alpha$, then for every $a \in \mathcal{U}_\beta$ there is $b \in \mathcal{U}_\beta$ such that $b \subset^* a$ and that there are $\pi, \psi \in \omega^\omega$ and $d^* \leq_0 d$ so that $\langle\pi, \psi, d^*\rangle$ is a normal triple, $\pi'' \text{set}(d^*) = b$ and $\forall k \in \text{set}(d^*)\ [\pi_1(k) = \pi_{\beta,\alpha}(\pi(k))]$.

[2] $\mathcal{U}$ is rapid if for every $f \in \omega^\omega$ there is $X \in \mathcal{U}$ such that $X(n) \geq f(n)$ for each $n < \omega$

δ-generic sequence of P-points

- if $\alpha < \beta < \delta$, then $\mathcal{U}_\beta \not\leq_T \mathcal{U}_\alpha$.
- for every $\alpha < \delta$, $\pi_{\alpha,\alpha} = \mathrm{id}$ and:
  - $\forall \alpha \leq \beta < \delta \ \forall i < \mathfrak{c} \ [\pi''_{\beta,\alpha} \operatorname{set}(c_i^\beta) \in \mathcal{U}_\alpha]$;
  - $\forall \alpha \leq \beta \leq \gamma < \delta \ \exists i < \mathfrak{c} \ \forall^\infty k \in \operatorname{set}(c_i^\gamma) \ [\pi_{\gamma,\alpha}(k) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(k))]$;
  - for $\alpha < \beta < \delta$ there are $i < \mathfrak{c}$, $b_{\beta,\alpha} \in \mathbb{P}$ and $\psi_{\beta,\alpha} \in \omega^\omega$ such that $\langle \pi_{\beta,\alpha}, \psi_{\beta,\alpha}, b_{\beta,\alpha} \rangle$ is a normal triple and $c_i^\beta \leq b_{\beta,\alpha}$;

δ-generic sequence of P-points

- if $\mu < \delta$ is a limit ordinal such that $\mathrm{cof}(\mu) = \omega$, $X \subset \mu$ is a countable set such that $\sup(X) = \mu$, $\langle d_j : j < \omega \rangle$ is a decreasing sequence of conditions in $\mathbb{P}$, and $\langle \pi_\alpha : \alpha \in X \rangle$ is a sequence of maps in ω^ω such that:
  - $\forall \alpha \in X \ \forall j < \omega \ [\pi''_\alpha \, \mathrm{set}(d_j) \in \mathcal{U}_\alpha]$;
  - $\forall \alpha, \beta \in X \ [\alpha \leq \beta \Rightarrow \exists j < \omega \ \forall^\infty k \in \mathrm{set}(d_j) \ [\pi_\alpha(k) = \pi_{\beta,\alpha}(\pi_\beta(k))]]$;
  - for all $\alpha \in X$ there are $j < \omega$, $b_\alpha \in \mathbb{P}$ and $\psi_\alpha \in \omega^\omega$ such that $\langle \pi_\alpha, \psi_\alpha, b_\alpha \rangle$ is a normal triple and $d_j \leq b_\alpha$;

  then the set of all $i^* < \mathfrak{c}$ such that there are $d^* \in \mathbb{P}$ and $\pi, \psi \in \omega^\omega$ satisfying:
  - $\forall j < \omega \ [d^* \leq d_j]$ and $\mathrm{set}(c^\mu_{i^*}) = \pi'' \, \mathrm{set}(d^*)$;
  - $\forall \alpha \in X \ \forall^\infty k \in \mathrm{set}(d^*) \ [\pi_\alpha(k) = \pi_{\mu,\alpha}(\pi(k))]$;
  - $\langle \pi, \psi, d^* \rangle$ is a normal triple;

  is cofinal in $\mathfrak{c}$;

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Which partial orders can be embedded into $\langle \mathcal{R}, \leq_{RK} \rangle$ under suitable hypothesis?

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- Such a poset must be of cardinality at most $2^{\mathfrak{c}}$;
- Such a poset must be locally $\mathfrak{c}$, i.e. each element can have only $\mathfrak{c}$ many predecessors.
- It is plausible that these are the only two obstacles.