Between Hausdorff and Selective

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- For example $Fin = [\omega]^{<\omega}$ is an ideal itself.

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- The ultrapower of X with respect to $\mathcal{U}$ is $Ult_{\mathcal{U}}(X) = X^\omega / \sim$.

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- The topology of the ultrapower is $\tau^* = \langle \{ U^* : U \in \tau \} \rangle$.

- Let $\mathcal{U}$ be an ultrafilter on ω and X an infinite set, we say that $\mathcal{U}$ is (X, τ)-Hausdorff if and only if $(Ult_{\mathcal{U}}(X), \tau^*)$ is a topological Hausdorff space.

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- We just say that $\mathcal{U}$ is Hausdorff if it is $(\omega, \mathcal{P}(\omega))$-Hausdorff. This is the usual definition of Hausdorff ultrafilter.

Some easy observations

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- In particular, if $\mathcal{U}$ is (X, τ)-Hausdorff, then $\mathcal{U}$ is just Hausdorff, because the discrete topology contains all the other topologies.

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- We say that $\mathcal{U}$ is $[\omega]^{<\omega}$-Hausdorff if it is $([\omega]^{<\omega}, \tau)$-Hausdorff.

Proposition

Suppose that $\mathcal{U}$ is an ultrafilter on ω. $\mathcal{U}$ is $[\omega]^{<\omega}$-Hausdorff if and only if for every sequence $(x_n \in [\omega]^{<\omega} \setminus \{\emptyset\} : n \in \omega)$ there is $A \subseteq \omega$ such that $\{n : |A \cap x_n| \equiv 1 \bmod 2\} \in \mathcal{U}$.

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Proof.

The proof is just to translate the definitions. The good thing about this easy proposition is that we can forget the definition of being $[\omega]^{<\omega}$-Hausdorff and just use this combinatorial property as definition. $\square$

We have seen that $\mathcal{U}$ is $[\omega]^{<\omega}$-Hausdorff implies $\mathcal{U}$ is Hausdorff and it's not hard to see that every Ramsey ultrafilter is $[\omega]^{<\omega}$-Hausdorff, but we can go a little bit far.

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Cut and choose kind game

Given $\mathcal{I}$ an ideal on ω the game $G(\mathcal{I})$ is between two players. First player I makes a partition of ω in ω many pieces $\omega = \bigcup A_n$. In the n-th move player I cuts A_n into two pieces: A_n^0 and A_n^1 and player II chooses $i_n \in 2$. Finally, player I wins iff $\bigcup A_n^{i_n} \in \mathcal{I}$.

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- Player *II* has a winning stratagy in the game $G(\mathcal{U}^*)$ $\Rightarrow$
- Player *I* does not have a winning strategy in $G(\mathcal{U}^*)$ $\Rightarrow$
- $\mathcal{U}$ is $[\omega]^{<\omega}$-Hausdorff $\Rightarrow$
- $\mathcal{U}$ is Hausdorff

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- Our matter is: what happens in the middle?

- We will see that consistently any of the implications is not an equivalence. We will not see any complete proof because any of them are a little bit long, but we will see the ideas.

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Fact 1

The generic filter $\mathcal{U}_{gen}$ for the forcing $(\mathcal{P}(\omega)/\mathcal{I}, \subseteq_{\mathcal{I}})$ can be seen as an ultrafilter on ω and $\mathcal{I} \cap \mathcal{U}_{gen} = \emptyset$.

Proposition

It is consistent with ZFC that there is $\mathcal{U}$ an ultrafilter on ω such that player *II* has a winning strategy in $G(\mathcal{U}^*)$ but $\mathcal{U}$ is not selective.

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Proof.

Consider the $\mathcal{ED}_{fin}$ ideal. This is an F_σ ideal and $\mathcal{R} \leq_K \mathcal{ED}_{fin}$. Suppose that $\mathcal{U}_{gen}$ is the generic ultrafilter when we force with $\mathcal{P}(\omega)/\mathcal{ED}_{fin}$, then player *II* has a winning strategy in $G(\mathcal{U}^*_{gen})$ and $\mathcal{R} \leq \mathcal{U}^*_{gen}$ so $\mathcal{U}_{gen}$ can't be selective. $\square$

Proposition

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Proof.

Now we have to consider the $\mathcal{I}_{\frac{1}{n}}$ ideal. Again this is an F_σ ideal so we have $\mathcal{U}_{gen}$ the generic ultrafilter when we force with $\mathcal{P}(\omega)/\mathcal{ED}_{fin}$, this ultrafilter is that we are looking for, basically because by genericity we can do "big" or "small" almost any positive set, bellow any other positive set (condition). $\square$

Proposition

It is consistent with ZFC that there is $\mathcal{U}$ an ultrafilter on ω such that $\mathcal{U}$ is $[\omega]^{<\omega}$-Hausdorff but player I has a winning strategy in $G(\mathcal{U}^*)$.

Proof.

The idea is the same as before, but we will construct an ad-hoc F_σ ideal. The ideal will be deffined in $\omega \times \omega$. For $n \in \omega$, consider $\{A_\sigma \subset \{n\} \times \omega : \sigma \in 2^n\}$ a family of independent sets such that $A_{\tau^\frown 0}$ is the complement (in $\{n\} \times \omega$) of $A_{\tau^\frown 1}$.
The ideal $\mathcal{IN}$ is the ideal generated by $\{A \subseteq \omega \times \omega : (\exists x \in 2^\omega)(\forall n \in \omega)(A \cap \{n\} \times \omega \subseteq A_{x \restriction n})\}$. The forcing of $\mathcal{P}(\omega)/\mathcal{IN}$ ordered by contention will give us a generic ultrafilter as desired. □

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Proof.

Now the F_σ ideal will be deffined in $X = [\omega]^{<\omega} \setminus \{\emptyset\}$.
$I_A = \{a \in X : |A \cap a| \equiv 1 \text{ mod } 2\}$ for $A \subseteq \omega$. The ideal $\mathcal{HOM}$ is the ideal generated by $\{I_A : A \in \mathcal{P}(\omega)\}$ and by the combinatorial caracterization the generic ultrafilter could not be $[\omega]^{\omega}$-Hausdorff, but it is Hausdorff. □

- To finish the talk I just want to say that the idea of construct some definable ideal and force with the quotient was very usefull. We think that it is consistent with ZFC that for every $\mathcal{U}$ an ultrafilter, player I has a winning strategy in $G(\mathcal{U}^*)$ and we are working in this, but it is still open.

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- In ZFC: is there an ultrafilter which is $[\omega]^{<\omega}$-Hausdorff?

- Just as a comment: the game is very interesting, for example player *II* wins the game for almost all Borel ideals, but almost never for ultrafilters (and very likely it is consistent that never).

$\mathcal{THANK\ YOU\ FOR\ YOUR\ ATTENTION}$