Between Hausdorff and Selective

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1. Hausdorff ultrafilters

2. $[\omega]^{<\omega}$-Hausdorff ultrafilters

3. A cut and choose kind game

4. The main dish
We will study properties of filters and ideals on $\omega$. 
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For example $\text{Fin} = [\omega]^{<\omega}$ is an ideal itself.
Given \( \mathcal{U} \) an ultrafilter on \( \omega \), the equivalence relation \( \sim \) on \( X^\omega \) is defined as \( f \sim g \) if and only if \( \{ n : f(n) = g(n) \} \in \mathcal{U} \).
Given $\mathcal{U}$ an ultrafilter on $\omega$, the equivalence relation $\sim$ on $X^\omega$ is defined as $f \sim g$ if and only if $\{n : f(n) = g(n)\} \in \mathcal{U}$.

The ultrapower of $X$ with respect to $\mathcal{U}$ is $\text{Ult}_{\mathcal{U}}(X) = X^\omega / \sim$. 
Suppose that $\tau$ is a topology on $X$, and $U \in \tau$. Then $U^*$ is (by definition) $\{[f] : f \in X^\omega \land (\exists A \in U)(f[A] \subseteq U)\}$. 
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The topology of the ultrapower is $\tau^* = \langle\{U^* : U \in \tau\}\rangle$. 
Let $\mathcal{U}$ be an ultrafilter on $\omega$ and $X$ an infinite set, we say that $\mathcal{U}$ is $(X, \tau)$-Hausdorff if and only if $(\text{Ult}_\mathcal{U}(X), \tau^*)$ is a topological Hausdorff space.
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We just say that $\mathcal{U}$ is Hausdorff if it is $(\omega, \mathcal{P}(\omega))$-Hausdorff. This is the usual definition of Hausdorff ultrafilter.
Some easy observations

- Let $\mathcal{U}$ be an ultrafilter on $\omega$, if $\mathcal{U}$ is Ramsey then $\mathcal{U}$ is Hausdorff.
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2. If $\tau \subseteq \sigma$ are topologies of $X$, and $\mathcal{U}$ is an ultrafilter (on $\omega$) then $\mathcal{U}$ is $(X, \tau)$-Hausdorff implies that $\mathcal{U}$ is $(X, \sigma)$-Hausdorff.
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- If $\tau \subseteq \sigma$ are topologies of $X$, and $\mathcal{U}$ is an ultrafilter (on $\omega$) then $\mathcal{U}$ is $(X, \tau)$-Hausdorff implies that $\mathcal{U}$ is $(X, \sigma)$-Hausdorff.

- In particular, if $\mathcal{U}$ is $(X, \tau)$-Hausdorff, then $\mathcal{U}$ is just Hausdorff, because the discrete topology contains all the other topologies.
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Consider \(\tau\) the minimum topology which makes continuous all the homomorphisms from \([\omega]^{<\omega}\) to 2. Obviously \(([\omega]^{<\omega}, \tau)\) is a topological group.

We say that \(\mathcal{U}\) is \([\omega]^{<\omega}\)-Hausdorff if it is \(([\omega]^{<\omega}, \tau)\)-Hausdorff.
Proposition

Suppose that $\mathcal{U}$ is an ultrafilter on $\omega$. $\mathcal{U}$ is $[\omega]^{<\omega}$-Hausdorff if and only if for every sequence $(x_n \in [\omega]^{<\omega} \setminus \{\emptyset\} : n \in \omega)$ there is $A \subseteq \omega$ such that $\{n : |A \cap x_n| \equiv 1 \mod 2\} \in \mathcal{U}$. 
Proposition

Suppose that $\mathcal{U}$ is an ultrafilter on $\omega$. $\mathcal{U}$ is $[\omega]^\omega$-Hausdorff if and only if for every sequence $(x_n \in [\omega]^\omega \setminus \{\emptyset\} : n \in \omega)$ there is $A \subseteq \omega$ such that
$$\{n : |A \cap x_n| \equiv 1 \mod 2\} \in \mathcal{U}.$$ 

Proof.

The proof is just to translate the definitions. The good thing about this easy proposition is that we can forget the definition of being $[\omega]^\omega$-Hausdorff and just use this combinatorial property as definition. 

□
We have seen that $\mathcal{U}$ is $[\omega]^\omega$-Hausdorff implies $\mathcal{U}$ is Hausdorff and it’s not hard to see that every Ramsey ultrafilter is $[\omega]^\omega$-Hausdorff, but we can go a little bit far.
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Cut and choose kind game

Given $\mathcal{I}$ an ideal on $\omega$ the game $G(\mathcal{I})$ is between two players. First player $I$ makes a partition of $\omega$ in $\omega$ many pieces $\omega = \bigcup A_n$. In the $n$-th move player $I$ cuts $A_n$ into two pieces: $A^0_n$ and $A^1_n$ and player $II$ chooses $i_n \in 2$. Finally, player $I$ wins iff $\bigcup A^i_n \in \mathcal{I}$. 
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- $\mathcal{U}$ is selective $\Rightarrow$
- Player II has a winning strategy in the game $G(\mathcal{U}^*) \Rightarrow$
- Player I does not have a winning strategy in $G(\mathcal{U}^*) \Rightarrow$
Proposition

Let \( U \) be an ultrafilter on \( \omega \). We have that:

- \( U \) is selective \( \Rightarrow \)
- \( \) Player \( II \) has a winning stratagy in the game \( G(U^*) \) \( \Rightarrow \)
- \( \) Player \( I \) does not have a winning strategy in \( G(U^*) \) \( \Rightarrow \)
- \( U \) is \( [\omega]^{<\omega} \)-Hausdorff \( \Rightarrow \)
Proposition

Let $\mathcal{U}$ be an ultrafilter on $\omega$. We have that:

- $\mathcal{U}$ is selective $\Rightarrow$
- Player II has a winning strategy in the game $G(\mathcal{U}^*)$ $\Rightarrow$
- Player I does not have a winning strategy in $G(\mathcal{U}^*)$ $\Rightarrow$
- $\mathcal{U}$ is $[\omega]<\omega$-Hausdorff $\Rightarrow$
- $\mathcal{U}$ is Hausdorff
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Our matter is: what happens in the middle?

We will see that consistently any of the implications is not an equivalence. We will not see any complete proof because any of them are a little bit long, but we will see the ideas.
Suppose that $\mathcal{I}$ is an $F_\sigma$ ideal on $\omega$. 
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**Fact 0**

The forcing $(\mathcal{P}(\omega)/\mathcal{I}, \subseteq \mathcal{I})$ is $\sigma$-closed, in particular it does not add reals.
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**Fact 1**
The generic filter $\mathcal{U}_{gen}$ for the forcing $(\mathcal{P}(\omega)/\mathcal{I}, \subseteq\mathcal{I})$ can be seen as an ultrafilter on $\omega$ and $\mathcal{I} \cap \mathcal{U}_{gen} = \emptyset$. 
Proposition

It is consistent with ZFC that there is $\mathcal{U}$ an ultrafilter on $\omega$ such that player II has a winning strategy in $G(\mathcal{U}^*)$ but $\mathcal{U}$ is not selective.
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It is consistent with ZFC that there is \( \mathcal{U} \) an ultrafilter on \( \omega \) such that player II has a winning strategy in \( G(\mathcal{U}^*) \) but \( \mathcal{U} \) is not selective.

Proof.

Consider the \( \mathcal{E}D_{\text{fin}} \) ideal. This is an \( F_\sigma \) ideal and \( \mathcal{R} \leq_{K} \mathcal{E}D_{\text{fin}} \). Suppose that \( \mathcal{U}_{\text{gen}} \) is the generic ultrafilter when we force with \( \mathcal{P}(\omega)/\mathcal{E}D_{\text{fin}} \), then player II has a winning strategy in \( G(\mathcal{U}_{\text{gen}}^*) \) and \( \mathcal{R} \leq \mathcal{U}_{\text{gen}}^* \) so \( \mathcal{U}_{\text{gen}} \) can’t be selective.
Proposition

It is consistent with ZFC that there is $\mathcal{U}$ an ultrafilter on $\omega$ such that none of players has a winning strategy in $G(\mathcal{U}^*)$. 
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Proof.
Now we have to consider the $\mathcal{I}_{1\over n}$ ideal. Again this is an $F_\sigma$ ideal so we have $\mathcal{U}_{gen}$ the generic ultrafilter when we force with $\mathcal{P}(\omega)/\mathcal{E}D_{fin}$, this ultrafilter is that we are looking for, basically because by genericity we can do "big" or "small" almost any positive set, below any other positive set (condition).
Proposition

It is consistent with ZFC that there is $\mathcal{U}$ an ultrafilter on $\omega$ such that $\mathcal{U}$ is $[\omega]^{<\omega}$-Hausdorff but player $I$ has a winning strategy in $G(\mathcal{U}^*)$. 
Proof.

The idea is the same as before, but we will construct an ad-hoc $F_\sigma$ ideal. The ideal will be defined in $\omega \times \omega$. For $n \in \omega$, consider
\[ \{ A_\sigma \subseteq \{n\} \times \omega : \sigma \in 2^n \} \]
a family of independent sets such that $A_{\tau \downarrow 0}$ is the complement (in $\{n\} \times \omega$) of $A_{\tau \downarrow 1}$.

The ideal $\mathcal{I}\mathcal{N}$ is the ideal generated by
\[ \{ A \subseteq \omega \times \omega : (\exists x \in 2^\omega)(\forall n \in \omega)(A \cap \{n\} \times \omega \subseteq A_{x|n}) \} \].

The forcing of $\mathcal{P}(\omega)/\mathcal{I}\mathcal{N}$ ordered by contention will give us a generic ultrafilter as desired.
Proposition

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Proof.

Now the $F_{\sigma}$ ideal will be defined in $X = [\omega]^{<\omega} \setminus \{\emptyset\}$.

$I_A = \{a \in X : |A \cap a| \equiv 1 \mod 2\}$ for $A \subseteq \omega$. The ideal $\mathcal{HOM}$ is the ideal generated by $\{I_A : A \in \mathcal{P}(\omega)\}$ and by the combinatorial characterization the generic ultrafilter could not be $[\omega]^{\omega}$-Hausdorff, but it is Hausdorff. \qed
To finish the talk I just want to say that the idea of constructing some definable ideal and force with the quotient was very useful. We think that it is consistent with ZFC that for every ultrafilter $\mathcal{U}$, player $I$ has a winning strategy in $G(\mathcal{U}^*)$ and we are working in this, but it is still open.
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In ZFC: is there an ultrafilter which is $[\omega]^{<\omega}$-Hausdorff?
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In ZFC: is there an ultrafilter which is $[\omega]^{<\omega}$-Hausdorff?

Just as a comment: the game is very interesting, for example player II wins the game for almost all Borel ideals, but almost never for ultrafilters (and very likely it is consistent that never).
THANK YOU FOR YOUR ATTENTION