Every $\mathcal{M}$-additive set is $\mathcal{E}$-additive: application of fractal dimensions

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Additive properties

Objects: $X \subseteq 2^\omega$

Adding sets: $X + Y = \{x + y : x \in X,\ y \in Y\}$

(addition coordinatewise modulo 2)

Measure on 2^ω: The usual product measure

Metric on 2^ω: $d(x, y) = 2^{-n(x,y)}$

Definitions

$\mathcal{N}^*$	**$\mathcal{N}$-additive:**	$X + N \in \mathcal{N}$ for each $N \in \mathcal{N}$
$\mathcal{M}^*$	**$\mathcal{M}$-additive:**	$X + M \in \mathcal{M}$ for each $M \in \mathcal{M}$
$\mathcal{E}^*$	**$\mathcal{E}$-additive:**	$X + E \in \mathcal{E}$ for each $E \in \mathcal{E}$
$\mathcal{SN}$	**strongly null:**	$X + M \neq 2^\omega$ for each $M \in \mathcal{M}$

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Inclusions

$\mathcal{N}^*$

Theorem (Shelah 1995)

$\mathcal{N}^* \implies \mathcal{M}^*$

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Theorem (Corollary to Pawlikowski 1995)

$\mathcal{E}^* \implies \mathcal{SN}$

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$\Downarrow$

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(T')

$\mathcal{M}^*$

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Nowik, Weiss 2002:

Definition (DON'T READ IT!)

X is (T') if: $\exists g \in \omega^\omega\ \forall f \in \omega^{\uparrow\omega}\ \exists I \in [\omega]^\omega\ \exists \langle H_n : n \in I\rangle$

1. $\forall n \in I\ H_n \subseteq 2^{[f(n),f(n+1))}$,
2. $\forall n \in I\ |H_n| \leqslant g(n)$,
3. $X \subseteq \{x \in 2^\omega : \forall^\infty n \in I\ x{\restriction}[f(n), f(n+1)) \in H_n\}$.

Theorem

$\mathcal{N}^* \implies (\mathrm{T}') \implies \mathcal{M}^*$

Question

$\mathcal{E}^* \iff (\mathrm{T}')$???

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Theorem (Galvin–Mycielski–Solovay)

X is $\mathcal{SN}$ if, and only if:
For any sequence $\varepsilon_n > 0$ there is a cover $\{U_n\}$ of X such that $\operatorname{diam} U_n < \varepsilon_n$.

- $\dim_H X = 0$

X is $\mathcal{H}$-null $\stackrel{\text{def}}{\equiv}$ $\dim_H f(X) = 0$ for all uniformly continuous f.

The following are equivalent:

- *X is $\mathcal{SN}$*
- *X is $\mathcal{H}$-null*
- *$\mathcal{H}^h(X) = 0$ for each Hausdorff function h*

$\mathcal{H}$-null sets

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Upper Hausdorff dimension

Definition (Upper Hausdorff dimension)

$$\overline{\dim}_{\mathsf{H}} X = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\} = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\}$$

Upper Hausdorff measure:

- $\overline{\mathcal{H}}^s_0(X) = \sup_{\delta>0} \inf\{\sum_{i=1}^n (dE_n)^s : d(E_i) \leqslant \delta,\ X \subseteq \underbrace{E_1 \cup \dots \cup E_n}_{\text{finite covers!}}\}$
- $\overline{\mathcal{H}}^s(X) = \inf\{\sum_{n=1}^\infty \overline{\mathcal{H}}^s_0(X_i) : X \subseteq X_1 \cup X_2 \cup \dots\}$ (Method I)

Elementary facts:

- If X is σ-compact, then $\overline{\dim}_{\mathsf{H}} X = \dim_{\mathsf{H}} X$
- If $Y \supseteq X$ is complete, then

$$\overline{\dim}_{\mathsf{H}} X = \inf\{\dim_{\mathsf{H}} K : X \subseteq K \subseteq Y,\ K\ \sigma\text{-compact}\}$$

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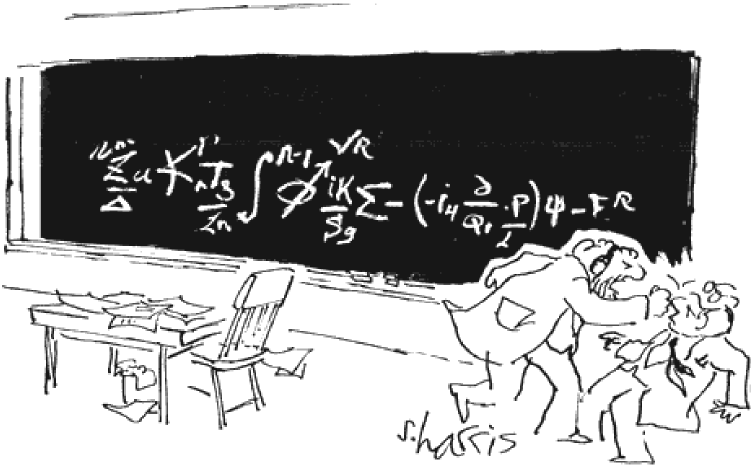
$\overline{\mathcal{H}}$-null sets and products

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\forall E \in \mathcal{E} \quad \overline{\mathcal{H}}^1(X \times E) = 0$
- $\forall E \in \mathcal{E}\ \exists K \supseteq X$ *σ-compact* $\quad \overline{\mathcal{H}}^1(X \times E) = 0$

Want proof?



"You want proof? I'll give you proof!"

A courtesy of Mr. Harris ©ScienceCartoonsPlus.com

$\overline{\mathcal{H}}$-null sets and products

Lemma

The following are equivalent:

- $\overline{\mathcal{H}}^h(X) = 0$ *for each Hausdorff function* h
- $\overline{\mathcal{H}}^1(X \times E) = 0$ *for each* $E \in \mathcal{E}$

$\Downarrow$ Assume X is $\mathcal{H}$-null

- $E \in \mathcal{E} \implies \mathcal{P}^g(E) = 0$ for some $g \prec 1$ [$g(r)$ grows faster than r]
- There is h such that $gh \succcurlyeq 1$
- Howroyd formula: $\overline{\mathcal{H}}^1(X \times E) \leqslant \overline{\mathcal{H}}^{gh}(X \times E) \leqslant \overline{\mathcal{H}}^h(X) \cdot \mathcal{P}^g(E) = 0$

$\Uparrow$ Assume X is not $\mathcal{H}$-null

- There is h such that $\overline{\mathcal{H}}^h(X) > 0$
- There is $g \prec 1$ such that $gh \preccurlyeq 1$
- Find $E \in \mathcal{E}$ such that $\mathcal{H}^g E) > 0$
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Strongly additive properties

Definition

- X is **strongly $\mathcal{M}$-additive** ($\mathcal{M}^\sharp$) if

$$\forall M \in \mathcal{M}\ \exists K \supseteq X\ \sigma\text{-compact} \quad K + M \in \mathcal{M}$$

- X is **strongly $\mathcal{E}$-additive** ($\mathcal{E}^\sharp$)

$$\forall N \in \mathcal{N}\ \exists K \supseteq X\ \sigma\text{-compact} \quad K + N \in \mathcal{N}$$

- X is **strongly strongly null** ($\mathcal{SN}^\sharp$) if

$$\forall M \in \mathcal{M}\ \exists K \supseteq X\ \sigma\text{-compact} \quad K + M \neq 2^\omega$$

Theorem

$\overline{\mathcal{H}}$-null $\iff$ $\mathcal{M}^*$ $\iff$ $\mathcal{M}^\sharp$ $\iff$ $\mathcal{E}^\sharp$ $\iff$ $\mathcal{SN}^\sharp$

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Lemma

$\overline{\mathcal{H}}$-null $\implies \mathcal{E}^\sharp$

Proof.

Fix $E \in \mathcal{E}$.

- There is $K \supseteq X$ σ-compact such that $\mathcal{H}^1(K \times E) = 0$
- $(x, y) \mapsto x + y$ is Lipschitz
- Thus $\overline{\mathcal{H}}^1(K + E) = 0$, i.e. $K + E \in \mathcal{E}$.

Lemma

$\mathcal{E}^\sharp \implies \mathcal{SN}^\sharp$

Proof.

Fix $M \in \mathcal{M}$.

- **Pawlikowski 1995:** There is $E \in \mathcal{E}$ such that $K + E \in \mathcal{N} \implies K + M \neq 2^\omega$
- There is $K \supseteq X$ σ-compact such that $K + E \in \mathcal{E} \subseteq \mathcal{N}$
- Thus $K + M \neq 2^\omega$

$\overline{\mathcal{H}}$-null $\implies \mathcal{E}^\sharp \implies \mathcal{SN}^\sharp \implies \mathcal{M}^\sharp \implies \mathcal{M}^* \implies$ $\overline{\mathcal{H}}$-null

Lemma

$\mathcal{SN}^\sharp \implies \mathcal{M}^\sharp$

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Lemma

$\mathcal{M}^* \implies \overline{\mathcal{H}}$-null

Theorem (Shelah 1995 DON'T READ IT!)

If $X \subseteq 2^\omega$ is meager-additive, then:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m\ \exists k \in \omega$$

$$g(n) \leqslant f(k) < g(n+1)\ \&\ x{\restriction}\big[f(k), f(k+1)\big) = y{\restriction}\big[f(k), f(k+1)\big)$$

Proof – vague outline.

Fix $h \in \mathbb{H}$.

- Understand the condition: The balls

$$B(y, 2^{-f(k+1)}) + p, \qquad n \in \omega,\ g(n) \leqslant k < g(n+1),\ p \in 2^{f(k)}$$

  form the right cover of X.
- Define properly f.
- Calculate Hausdorff sums.

Lemma

$\mathcal{M}^* \Longrightarrow \overline{\mathcal{H}}$-null

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Consequences

Corollary

- $\mathcal{M}^* \iff \overline{\mathcal{H}}$-null
- *If X is $\mathcal{M}^*$ and $f : 2^\omega \to 2^\omega$, then $f(X)$ is $\mathcal{M}^*$.*

Corollary

$\mathcal{M}^* \implies \mathcal{E}^*$

$$\mathcal{N}^* \implies (\mathrm{T}') \implies \mathcal{M}^* \implies \mathcal{E}^* \implies \mathcal{SN}$$

Corollary

(CH) $\mathcal{E}^* \not\Rightarrow (\mathrm{T}')$

Consequences

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- $\mathcal{E}^*$: $\overline{\mathcal{H}}^1(X + E) = 0$ for all $E \in \mathcal{E}$

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$\mathcal{E}^* \iff \mathcal{M}^*$???

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The line

$$T : \begin{cases} 2^\omega \longrightarrow [0,1] \\ x \mapsto \frac{1}{2} \sum 2^{-n} x(n) \end{cases}$$

Proposition

- X *is* $\overline{\mathcal{H}}$-null $\iff T(X)$ *is* $\overline{\mathcal{H}}$-null
- (Weiss 2009) X *is* $\mathcal{M}^*$ $\iff T(X)$ *is* $\mathcal{M}^*$

Theorem ($X \subseteq \mathbb{R}$)

$\overline{\mathcal{H}}$-null $\iff \mathcal{M}^* \iff \mathcal{M}^\sharp \implies \mathcal{E}^\sharp \implies \mathcal{E}^*$

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Products

Theorem

- $\overline{\mathcal{H}}$-null $\times$ $\overline{\mathcal{H}}$-null *is* $\overline{\mathcal{H}}$-null
- $\overline{\mathcal{H}}$-null $\times$ $\mathcal{H}$-null *is* $\mathcal{H}$-null [Strengthens Scheepers' Theorem]

Corollary

- $X, Y \subseteq \mathbb{R}$ *are* $\mathcal{M}^*$ $\implies$ $X \times Y$ *is* $\mathcal{M}^*$
- $X \subseteq \mathbb{R}^n$ *is* $\mathcal{M}^*$ $\iff$ *all projections of* X *are* $\mathcal{M}^*$

Products

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- $\overline{\mathcal{H}}$-null $\times$ $\mathcal{H}$-null *is* $\mathcal{H}$-null [Strengthens Scheepers' Theorem]

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Packing dimensions

- Hausdorff dimension $\dim_{\mathsf{H}} X$ … $\mathcal{H}$-null
- Upper Hausdorff dimension $\overline{\dim}_{\mathsf{H}} X$ … $\overline{\mathcal{H}}$-null
- Directed lower packing dimension $\underrightarrow{\dim}_{\mathsf{P}} X$ … $\underrightarrow{\mathcal{P}}$-null
- Upper packing dimension $\overline{\dim}_{\mathsf{P}} X$ … $\overline{\mathcal{P}}$-null

$$\overline{\dim}_{\mathsf{P}} X \geqslant \underrightarrow{\dim}_{\mathsf{P}} X \geqslant \overline{\dim}_{\mathsf{H}} X \geqslant \dim_{\mathsf{H}} X$$

Theorem

$\overline{\mathcal{P}}$-null	$\underrightarrow{\mathcal{P}}$-null	$\overline{\mathcal{H}}$-null	$\mathcal{H}$-null
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
$\mathcal{N}^*$	(T')	$\mathcal{M}^*$	$\mathcal{SN}$

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Theorem

$$\begin{array}{ccccccc} \overline{\mathcal{P}}\text{-null} & \Longrightarrow & \underrightarrow{\mathcal{P}}\text{-null} & \Longrightarrow & \overline{\mathcal{H}}\text{-null} & \Longrightarrow & \mathcal{H}\text{-null} \\ \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\ \mathcal{N}^* & \Longrightarrow & (\mathrm{T}') & \Longrightarrow & \mathcal{M}^* & \Longrightarrow & \mathcal{SN} \end{array}$$

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\end{array}$$

Definition

X is topologically $\mathcal{H}$-null $\stackrel{\text{def}}{\equiv}$ $\dim_{\mathsf{H}} f(X)$ for each continuous f.

Theorem

- *topologically* $\mathcal{H}$-null $\iff$ *Rothberger property*
- *topologically* $\overline{\mathcal{H}}$-null $\iff$ *Gerlits–Nagy property*
- *topologically* $\mathcal{P}$-null $\Longleftarrow$ *strong* γ*-set*
- *but consistently topologically* $\mathcal{P}$-null $\not\Rightarrow$ *strong* γ*-set*

Topologically null sets

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That's all Folks!