Crookedness and almost homogeneity in categories of compacta

Adam Bartoš
drekin@gmail.com

Faculty of Mathematics and Physics, Charles University
Institute of Mathematics, Czech Academy of Sciences

Winter School in Abstract Analysis
Section Set Theory & Topology
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This is joint work with Wiesław Kubiś.
An inverse sequence \( \langle X_*, f_* \rangle \) of topological spaces and continuous maps, and its limit \( \langle X_\infty, f_{n,\infty} \rangle_{n \in \omega} \):

\[
X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots X_n \leftarrow X_{n+1} \leftarrow \cdots X_\infty
\]

\( f_{1,3} \quad f_{n,\infty} \)
An *inverse sequence* $\langle X_*, f_* \rangle$ of topological spaces and continuous maps, and its *limit* $\langle X_\infty, f_n, \infty \rangle_{n \in \omega}$:

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A *category of compacta* is any category $\mathcal{K}$ whose objects are metrizable compacta and whose morphisms are continuous maps. A *sequence in $\mathcal{K}$* is an inverse sequence of $\mathcal{K}$-objects and $\mathcal{K}$-maps.
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$\sigma\mathcal{K}$ denotes the category of all limits of sequences in $\mathcal{K}$ and all limits of almost transformations between sequences in $\mathcal{K}$. 

I denotes the category with the only object $I := [0, 1]$ and all continuous surjections.
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A category of compacta is any category \( \mathcal{K} \) whose objects are metrizable compacta and whose morphisms are continuous maps. A sequence in \( \mathcal{K} \) is an inverse sequence of \( \mathcal{K} \)-objects and \( \mathcal{K} \)-maps.

\( \sigma \mathcal{K} \) denotes the category of all limits of sequences in \( \mathcal{K} \) and all limits of almost transformations between sequences in \( \mathcal{K} \).

\( \mathcal{I} \) denotes the category with the only object \( I := [0, 1] \) and all continuous surjections.
A continuum $X$ is *arc-like* if it is the limit of a sequence in $\mathcal{I}$:

$$
\begin{array}{cccc}
& f_0 & f_1 & f_2 & f_3 & \cdots & X \\
\|& \leftarrow & \leftarrow & \leftarrow & \leftarrow & \cdots & \\
\end{array}
$$
Motivation

A continuum $X$ is *arc-like* if it is the limit of a sequence in $\mathcal{I}$:

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$$

So $\sigma\mathcal{I}$-objects are exactly the arc-like continua, and by [Mardešić–Segal, 1963], $\sigma\mathcal{I}$-maps are all continuous surjections.

Fact [Bing, 1951] There exists a unique hereditarily indecomposable arc-like continuum, called the *pseudo-arc*.

Fact [Irwin–Solecki, 2006] The pseudo-arc is the quotient induced by a topological model-theoretic projective Fraïssé limit.
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- A compact Hausdorff space is *hereditarily indecomposable* if for every subcontinua $C, D \subseteq X$ we have $C \subseteq D$ or $C \supseteq D$ or $C \cap D = \emptyset$.

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The pseudo-arc is the quotient induced by a topological model-theoretic projective *Fraïssé limit*. 
Bing’s result may be reproved using the following.

**Theorem**

Let $\langle X_*, f_* \rangle$ be a sequence in $\mathcal{I}$. The following conditions are equivalent.

1. $X_\infty$ is hereditarily indecomposable.
2. $X_\infty$ is *crooked*.
3. The maps $f_{n,\infty}$ are *crooked*.
4. $\langle X_*, f_* \rangle$ is a *crooked sequence*.
5. $\langle X_*, f_* \rangle$ is a *Fraïssé sequence*.
6. $X_\infty$ is *universal* and *almost projective* in $\sigma\mathcal{I}$.
7. $X_\infty$ is *universal* and *almost homogeneous* in $\sigma\mathcal{I}$.
“When going from $A$ to $B$, we first have to go from $A$ near $B$, then return near $A$, and finally go to $B$.”
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A map $f : \{0, 1, \ldots, m\} \rightarrow \{0, 1, \ldots, n\}$ is crooked if for every $i \leq j \leq m$ there are $i \leq j' \leq i' \leq j$ such that $|f(i) - f(i')| \leq 1$ and $|f(j) - f(j')| \leq 1$.

Fact:
For every $\varepsilon > 0$ there is an $\varepsilon$-crooked $I$-map (e.g. the maps $\sigma_n$ [Lewis–Minc, 2010]).
“When going from $A$ to $B$, we first have to go from $A$ near $B$, then return near $A$, and finally go to $B$.”

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A map $f: \mathbb{I} \rightarrow \mathbb{I}$ is $\varepsilon$-**crooked** if for every $x \leq y \in \mathbb{I}$ there are $x \leq y' \leq x' \leq y$ such that $f(x) \approx_\varepsilon f(x')$ and $f(y) \approx_\varepsilon f(y')$. 

**Fact** For every $\varepsilon > 0$ there is an $\varepsilon$-crooked $\mathbb{I}$-map (e.g., the maps $\sigma_n$ [Lewis–Minc, 2010]).
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A map $f : \mathbb{I} \rightarrow \mathbb{I}$ is $\varepsilon$-*crooked* if for every $x \leq y \in \mathbb{I}$ there are $x \leq y' \leq x' \leq y$ such that $f(x) \approx_{\varepsilon} f(x')$ and $f(y) \approx_{\varepsilon} f(y')$.

**Fact**

For every $\varepsilon > 0$ there is an $\varepsilon$-crooked $\mathcal{I}$-map (e.g. the maps $\sigma_n$ [Lewis–Minc, 2010]).
<table>
<thead>
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<td>Let $X$ be a topological space.</td>
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Let $X$ be a topological space.

- A quadruple $\langle A, B, U, V \rangle$ is *admissible in* $X$ if $A, B$ are disjoint closed subsets of $X$ and $U, V$ are their open neighborhoods.

- $X$ is *crooked at* $\langle A, B, U, V \rangle$ if there are closed sets $F_0, F_1, F_2 \subseteq X$ such that $A \subseteq F_0$, $B \subseteq F_2$, $F_0 \cup F_1 \cup F_2 = X$, $F_0 \cap F_1 \subseteq V$, $F_1 \cap F_2 \subseteq U$, $F_0 \cap F_2 = \emptyset$. 

**Theorem [Krasinkiewicz–Minc, 1977]**

A compact Hausdorff space $X$ is hereditarily indecomposable if and only if it is crooked.
Definition [Krasinkiewicz–Minc, 1977]

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- $X$ is *crooked* if it is crooked at every admissible quadruple.
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- $X$ is crooked if it is crooked at every admissible quadruple.

Theorem [Krasinkiewicz–Minc, 1977]

A compact Hausdorff space $X$ is hereditarily indecomposable if and only if it is crooked.
Definition [Maćkowiak, 1985]

Let $f : X \to Y$ be a continuous map, $\langle A, B, U, V \rangle$ admissible in $Y$.

- $f$ is \textit{crooked at} $\langle A, B, U, V \rangle$ if $X$ is crooked at $\langle f^{-1}[A], f^{-1}[B], f^{-1}[U], f^{-1}[V] \rangle$.
- $f$ is \textit{crooked} if it is crooked at every admissible quadruple in $Y$.

So crookedness of $X$ is crookedness of $\text{id}_X$. 

**Definition [Maćkowiak, 1985]**

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- $f$ is **crooked** if it is crooked at every admissible quadruple in $Y$.

So crookedness of $X$ is crookedness of $id_X$.

**Definition**

Let $f : X \to \langle Y, d \rangle$ be a continuous map, $A, B \subseteq Y$ closed disjoint, and $\varepsilon > 0$.

- $f$ is **$\varepsilon$-crooked at** $\langle A, B \rangle$ if it is crooked at $\langle A, B, A^\varepsilon, B^\varepsilon \rangle$.
- $f$ is **$\varepsilon$-crooked** if it is $\varepsilon$-crooked at every closed disjoint $\langle A, B \rangle$. 
Proposition

A continuous map $f: \mathbb{I} \to \langle X, d \rangle$ is $\varepsilon$-crooked if and only if it satisfies the classical definition: for every $x \leq y \in \mathbb{I}$ there are $x \leq y' \leq x' \leq y$ such that $f(x) \approx_{\varepsilon} f(x')$ and $f(y) \approx_{\varepsilon} f(y')$. 
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Let $\langle X_*, f_* \rangle$ be an inverse sequence of metrizable compacta and continuous maps. The following conditions are equivalent.

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3. The maps $f_{n,\infty}$ are crooked.
4. $\langle X_*, f_* \rangle$ is a crooked sequence, i.e. for every $n$ and $\varepsilon > 0$ there is $m \geq n$ such that $f_{n,m}$ is $\varepsilon$-crooked.
Crookedness

Proposition

A continuous map \( f : \mathbb{I} \to \langle X, d \rangle \) is \( \varepsilon \)-crooked if and only if it satisfies the classical definition: for every \( x \leq y \in \mathbb{I} \) there are \( x \leq y' \leq x' \leq y \) such that \( f(x) \approx_\varepsilon f(x') \) and \( f(y) \approx_\varepsilon f(y') \).

Theorem

Let \( \langle X_\ast, f_\ast \rangle \) be an inverse sequence of metrizable compacta and continuous maps. The following conditions are equivalent.

1. \( X_\infty \) is hereditarily indecomposable.
2. \( X_\infty \) is crooked.
3. The maps \( f_{n,\infty} \) are crooked.
4. \( \langle X_\ast, f_\ast \rangle \) is a crooked sequence, i.e. for every \( n \) and \( \varepsilon > 0 \) there is \( m \geq n \) such that \( f_{n,m} \) is \( \varepsilon \)-crooked.

For Peano continua, this was essentially proved by [Brown, 1960].
**Definition**

A category of compacta $\mathcal{K}$ has the *almost amalgamation property* if for every $\mathcal{K}$-maps $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ and every $\varepsilon > 0$ there are $\mathcal{K}$-maps $f': W \rightarrow X$ and $g': W \rightarrow Y$ such that $f \circ f' \approx \varepsilon g \circ g'$.
Definition

A category of compacta $\mathcal{K}$ has the *almost amalgamation property* if for every $\mathcal{K}$-maps $f: X \to Z$ and $g: Y \to Z$ and every $\varepsilon > 0$ there are $\mathcal{K}$-maps $f': W \to X$ and $g': W \to Y$ such that $f \circ f' \approx \varepsilon g \circ g'$.
**Definition**

A category of compacta $\mathcal{K}$ has the *almost amalgamation property* if for every $\mathcal{K}$-maps $f : X \to Z$ and $g : Y \to Z$ and every $\varepsilon > 0$, there are $\mathcal{K}$-maps $f' : W \to X$ and $g' : W \to Y$ such that $f \circ f' \approx \varepsilon g \circ g'$.
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\[
\begin{array}{c}
\varepsilon > 0 \\
X & \approx_\varepsilon & W \\
\downarrow & \approx_\varepsilon & \\
Z & \approx_\varepsilon & W \\
Y & \approx_\varepsilon & \\
g & \approx_\varepsilon & f' \\
g' & \approx_\varepsilon & f
\end{array}
\]
Fraïssé theory – almost amalgamation property

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A category of compacta $\mathcal{K}$ has the **almost amalgamation property** if for every $\mathcal{K}$-maps $f: X \to Z$ and $g: Y \to Z$ and every $\varepsilon > 0$ there are $\mathcal{K}$-maps $f': W \to X$ and $g': W \to Y$ such that $f \circ f' \approx_\varepsilon g \circ g'$.

$$
\begin{align*}
Z & \xleftarrow{g} Y \xrightarrow{f} X \\
& \quad \approx_\varepsilon W \\
W & \xleftarrow{g'} Y \xrightarrow{f'} X
\end{align*}
$$

**Fact**

The interval category $\mathcal{I}$ has the almost amalgamation property by the mountain climbing theorem.
Definition

A sequence \( \langle X_*, f_* \rangle \) in a category of compacta \( \mathcal{K} \) is

- *universal*
A sequence \( \langle X_*, f_* \rangle \) in a category of compacta \( \mathcal{K} \) is

- **universal** if for every \( \mathcal{K} \)-object \( Y \)

\[
\begin{align*}
X_0 & \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} X_4 \xleftarrow{f_4} \cdots \\
& \quad Y
\end{align*}
\]
Definition

A sequence \( \langle X_*, f_* \rangle \) in a category of compacta \( \mathcal{K} \) is

- *universal* if for every \( \mathcal{K} \)-object \( Y \) there is a \( \mathcal{K} \)-map \( g : X_n \to Y \);

\[
\begin{array}{ccccccc}
X_0 & \leftarrow & X_1 & \leftarrow & \cdots & \leftarrow & X_n & \leftarrow & \cdots \\
\downarrow & f_0 & & f_1 & & & f_n & \downarrow \\
Y & & & & & & \end{array}
\]

- *Fraïssé* if it is both universal and almost projective.
Fraïssé theory – sequences

**Definition**

A sequence $\langle X_*, f_* \rangle$ in a category of compacta $\mathcal{K}$ is

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  ![Diagram](image)

- *almost projective*
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  $$
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  f_0 & & & f_1 & & & \cdots & & & f_n & \\
  \end{array}
  $$

- **almost projective** if for every $\mathcal{K}$-maps $f : X_n \to Z$, $g : Y \to Z$ and $\varepsilon > 0$

  $$
  \begin{array}{cccccc}
  & X_0 & \leftarrow & X_1 & \leftarrow & \cdots & \leftarrow & X_n & \leftarrow & \cdots \\
  f_0 & & & f_1 & & & \cdots & & & f_n & \\
  \end{array}
  $$

  $$
  \begin{array}{cccc}
  \varepsilon > 0 & \leftarrow & Z & \leftarrow & Y \\
  f & & & g \\
  \end{array}
  $$

Fraïssé theory – sequences
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Definition

A sequence \( \langle X_*, f_* \rangle \) in a category of compacta \( \mathcal{K} \) is

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\[
\begin{array}{ccccccccc}
X_0 & f_0 & X_1 & f_1 & \cdots & X_n & f_n & \cdots \\
& & & & & & & \\
& & & & & & & g \\
& & & & & & & \cdots \\
& & & & & & & Y \\
\end{array}
\]

- **almost projective** if for every \( \mathcal{K} \)-maps \( f : X_n \to Z, g : Y \to Z \)
  and \( \varepsilon > 0 \) there is a \( \mathcal{K} \)-map \( h : X_m \to Y \) such that
  \( f \circ f_{n,m} \approx_\varepsilon g \circ h \);

\[
\begin{array}{ccccccccc}
X_0 & f_0 & X_1 & f_1 & \cdots & X_n & f_{n,m} & X_m & \cdots \\
& & & & & & & & & & \\
& & & & & & & & & & h \\
& & \varepsilon > 0 & & & & & & & & \\
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A sequence $\langle X_*, f_* \rangle$ in a category of compacta $\mathcal{K}$ is

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  $X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$

  $\downarrow g$

  $Y$

- **almost projective** if for every $\mathcal{K}$-maps $f : X_n \to Z$, $g : Y \to Z$ and $\varepsilon > 0$ there is a $\mathcal{K}$-map $h : X_m \to Y$ such that

  $f \circ f_{n,m} \approx_{\varepsilon} g \circ h$;

  $X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow X_m \leftarrow \cdots$

  $\downarrow f$

  $\varepsilon > 0$

  $Z \leftarrow Y$

- **Fraïssé** if it is both universal and almost projective.
Theorem

A category of compacta $\mathcal{K}$ has a Fraïssé sequence if and only if it

1. is *directed*,
2. has a countable *universal family of objects*,
3. has the almost amalgamation property.
A category of compacta $\mathcal{K}$ has a Fraïssé sequence if and only if it

1. is directed,
2. has a countable universal family of objects,
3. has the almost amalgamation property.

Hence, $\mathcal{I}$ has a Fraïssé sequence.
**Definition**

Let $\mathcal{K}$ be category of compacta and let $X$ be a compactum. The *Banach–Mazur game* $BM_{\mathcal{K}}(X)$ is defined as follows. Eve starts with a $\mathcal{K}$-map $f_0 : X_0 \leftarrow X_1$, Odd responds with a $\mathcal{K}$-map $f_1 : X_1 \leftarrow X_2$, and so on. The outcome of the play is the sequence $\langle X_\ast, f_\ast \rangle$, and Odd wins if $X_\infty \cong X$. The space $X$ is *generic over $\mathcal{K}$* if Odd has a winning strategy for $BM_{\mathcal{K}}(X)$. 

*Observation* The generic object over $\mathcal{K}$ is unique (if it exists).

*Theorem* The limit of a Fraïssé sequence in $\mathcal{K}$ is generic over $\mathcal{K}$. Therefore, the *Fraïssé limit* is unique.
Definition

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The generic object over $\mathcal{K}$ is unique (if it exists).
Fraïssé theory – games and uniqueness

**Definition**

Let $\mathcal{K}$ be category of compacta and let $X$ be a compactum. The *Banach–Mazur game* $BM_\mathcal{K}(X)$ is defined as follows. Eve starts with a $\mathcal{K}$-map $f_0 : X_0 \leftarrow X_1$, Odd responds with a $\mathcal{K}$-map $f_1 : X_1 \leftarrow X_2$, and so on. The outcome of the play is the sequence $\langle X_*, f_* \rangle$, and Odd wins if $X_\infty \cong X$. The space $X$ is *generic over $\mathcal{K}$* if Odd has a winning strategy for $BM_\mathcal{K}(X)$.

**Observation**

The generic object over $\mathcal{K}$ is unique (if it exists).

**Theorem**

The limit of a Fraïssé sequence in $\mathcal{K}$ is generic over $\mathcal{K}$. Therefore, the *Fraïssé limit* is unique.
Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. An $\mathcal{L}$-object $X$ is

- universal in $\langle \mathcal{K}, \mathcal{L} \rangle$
Fraïssé theory – large objects

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Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. An $\mathcal{L}$-object $X$ is

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**Definition**

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. An $\mathcal{L}$-object $X$ is

- *universal in* $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object $Y$ there is an $\mathcal{L}$-map $f: X \to Y$;

- *almost projective in* $\langle \mathcal{K}, \mathcal{L} \rangle$
Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. An $\mathcal{L}$-object $X$ is

- **universal in** $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object $Y$ there is an $\mathcal{L}$-map $f : X \to Y$;

- **almost projective in** $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map $f : X \to Z$, $\mathcal{K}$-map $g : Y \to Z$, and $\varepsilon > 0$

\[ f \approx \varepsilon \circ g \circ h \]

We say just "in $\mathcal{L}$" instead of "in $\langle \mathcal{L}, \mathcal{L} \rangle$".
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Fraïssé theory – large objects

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Definition
Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. We consider the following condition for $\langle \mathcal{K}, \mathcal{L} \rangle$. 

Fact
It follows from the result [Mardešić–Segal, 1963] that $\langle I, \sigma I \rangle$ satisfies (F).
Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. We consider the following condition for $\langle \mathcal{K}, \mathcal{L} \rangle$.

(F) For every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$, every $\mathcal{L}$-map $g : X_\infty \to Y$ to a $\mathcal{K}$-object $Y$, and every $\varepsilon > 0$

\[ X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \cdots X_\infty \]

\[ Y \xleftarrow{g} \]
Fraïssé theory – condition (F)

**Definition**

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. We consider the following condition for $\langle \mathcal{K}, \mathcal{L} \rangle$.

**$(F)$** For every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$, every $\mathcal{L}$-map $g : X_\infty \to Y$ to a $\mathcal{K}$-object $Y$, and every $\varepsilon > 0$ there is a $\mathcal{K}$-map $g' : X_n \to Y$ such that $g \simeq_{\varepsilon} g' \circ f_{n,\infty}$.

\[
\begin{array}{cccccc}
X_0 & \xleftarrow{f_0} & X_1 & \xleftarrow{f_1} & \cdots & \xleftarrow{f_{n,\infty}} & X_\infty \\
\downarrow{g'} & & \downarrow{\simeq_{\varepsilon}} & & \downarrow{g} & & \\
Y & & & & & & \\
\end{array}
\]
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**(F)** For every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$, every $\mathcal{L}$-map $g : X_\infty \to Y$ to a $\mathcal{K}$-object $Y$, and every $\varepsilon > 0$ there is a $\mathcal{K}$-map $g' : X_n \to Y$ such that $g \approx_\varepsilon g' \circ f_{n,\infty}$.

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\downarrow^{f_0} & & \downarrow^{f_1} & & \cdots & & \downarrow^{f_{n,\infty}} & & \downarrow^{g'} \\
Y & \approx_\varepsilon & Y & & \approx_\varepsilon & & Y & & Y \\
g' & & g & & & & & & \\
\end{array}
\]

Fact

It follows from the result [Mardešić–Segal, 1963] that $\langle \mathcal{I}, \sigma \mathcal{I} \rangle$ satisfies (F).
Theorem

Let $\mathcal{K}$ be a category of compacta such that all $\mathcal{K}$-maps are surjections, and $\langle \mathcal{K}, \sigma \mathcal{K} \rangle$ satisfies (F). Then the following conditions are equivalent.

1. $\langle X^*, f^* \rangle$ is a Fraïssé sequence in $\mathcal{K}$.
2. $X^\infty$ is universal and almost projective in $\langle \mathcal{K}, \sigma \mathcal{K} \rangle$.
3. $X^\infty$ is universal and almost homogeneous in $\langle \mathcal{K}, \sigma \mathcal{K} \rangle$.
4. $X^\infty$ is universal and almost projective in $\sigma \mathcal{K}$.
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Hence, there is a unique Fraïssé limit in $\sigma I$ satisfying all the conditions.
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Fraïssé theory

**Theorem**

Let $\mathcal{K}$ be a category of compacta such that all $\mathcal{K}$-maps are surjections, and $\langle \mathcal{K}, \sigma \mathcal{K} \rangle$ satisfies (F). Then the following conditions are equivalent.

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Hence, there is a unique Fraïssé limit in $\sigma \mathcal{I}$ satisfying all the conditions.
There is a Fraïssé sequence in $\mathcal{I}$, and its limit is the unique universal and almost homogeneous object in $\sigma \mathcal{I}$. Since there is an $\varepsilon$-crooked $\mathcal{I}$-map for every $\varepsilon > 0$, it follows that any Fraïssé sequence in $\mathcal{I}$ is crooked, and so its limit is hereditarily indecomposable. By the result of Bing, the Fraïssé limit of $\sigma \mathcal{I}$ is the pseudo-arc.
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By the result of Bing, the Fraïssé limit of $\sigma\mathcal{I}$ is the pseudo-arc.
For every $\mathcal{I}$-map $g$ and every $\epsilon > 0$ there is $\delta > 0$ such that for every $\delta$-crooked $\mathcal{I}$-map $f$ there is an $\mathcal{I}$-map $h$ such that $f \approx_\epsilon g \circ h$. 

\[ \begin{array}{c} \mathcal{I} \\ \epsilon \\
\end{array} \xleftarrow{f} \xrightarrow{\approx_\epsilon} \mathcal{I} \xleftarrow{g} \xrightarrow{\epsilon} \mathcal{I} \xleftarrow{h} \]
The pseudo-arc

**Theorem**

For every $I$-map $g$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $\delta$-crooked $I$-map $f$ there is an $I$-map $h$ such that $f \approx_{\varepsilon} g \circ h$.

**Corollary**

- The almost amalgamation property of $I$ follows from the fact that there is an $\varepsilon$-crooked $I$-map for every $\varepsilon > 0$. 
The pseudo-arc

**Theorem**

For every $\mathcal{I}$-map $g$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $\delta$-crooked $\mathcal{I}$-map $f$ there is an $\mathcal{I}$-map $h$ such that $f \approx_\varepsilon g \circ h$.

**Corollary**

- The almost amalgamation property of $\mathcal{I}$ follows from the fact that there is an $\varepsilon$-crooked $\mathcal{I}$-map for every $\varepsilon > 0$.
- Every crooked sequence in $\mathcal{I}$ is almost projective, and hence Fraïssé. Therefore, there is a unique hereditarily indecomposable arc-like continuum.
Theorem

Let $\langle X_\ast, f_\ast \rangle$ be a sequence in $\mathcal{I}$. The following conditions are equivalent.

1. $X_\infty$ is hereditarily indecomposable.
2. $X_\infty$ is crooked.
3. The maps $f_{n,\infty}$ are crooked.
4. $\langle X_\ast, f_\ast \rangle$ is a crooked sequence.
5. $\langle X_\ast, f_\ast \rangle$ is a Fraïssé sequence.
6. $X_\infty$ is universal and almost projective in $\sigma \mathcal{I}$.
7. $X_\infty$ is universal and almost homogeneous in $\sigma \mathcal{I}$. 

Thank you for your attention.
Conclusion

Theorem

Let $\langle X_, f_* \rangle$ be a sequence in $\mathcal{I}$. The following conditions are equivalent.

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Thank you for your attention.