Crookedness and almost homogeneity in categories of compacta

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This is joint work with Wiesław Kubiś.

Notation

- An *inverse sequence* $\langle X_*, f_* \rangle$ of topological spaces and continuous maps, and its *limit* $\langle X_\infty, f_{n,\infty} \rangle_{n\in\omega}$:

$$X_0 \xleftarrow{f_0} X_1 \underbrace{\xleftarrow{f_1} X_2 \xleftarrow{f_2}}_{f_{1,3}} X_3 \xleftarrow{f_3} \cdots X_n \underbrace{\xleftarrow{f_n} X_{n+1} \leftarrow \cdots}_{f_{n,\infty}} X_\infty$$

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- $\sigma\mathcal{K}$ denotes the category of all limits of sequences in $\mathcal{K}$ and all limits of *almost transformations* between sequences in $\mathcal{K}$.
- $\mathcal{I}$ denotes the category with the only object $\mathbb{I} := [0, 1]$ and all continuous surjections.

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Fact [Irwin–Solecki, 2006]

The pseudo-arc is the quotient induced by a topological model-theoretic projective *Fraïssé limit*.

Motivation

Bing's result may be reproved using the following.

Theorem

Let $\langle X_*, f_* \rangle$ be a sequence in $\mathcal{I}$. The following conditions are equivalent.

1. X_∞ is hereditarily indecomposable.
2. X_∞ is *crooked*.
3. The maps $f_{n,\infty}$ are *crooked*.
4. $\langle X_*, f_* \rangle$ is a *crooked sequence*.
5. $\langle X_*, f_* \rangle$ is a *Fraïssé sequence*.
6. X_∞ is *universal* and *almost projective* in $\sigma\mathcal{I}$.
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- A map $f\colon \mathbb{I} \to \mathbb{I}$ is ε-*crooked* if for every $x \leq y \in \mathbb{I}$ there are $x \leq y' \leq x' \leq y$ such that $f(x) \approx_\varepsilon f(x')$ and $f(y) \approx_\varepsilon f(y')$.

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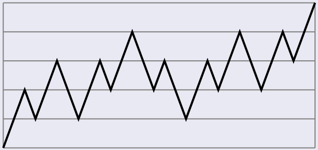
Fact

For every $\varepsilon > 0$ there is an ε-crooked $\mathcal{I}$-map (e.g. the maps σ_n [Lewis–Minc, 2010]).



Definition [Krasinkiewicz–Minc, 1977]

Let X be a topological space.

- A quadruple $\langle A, B, U, V \rangle$ is *admissible in X* if A, B are disjoint closed subsets of X and U, V are their open neighborhoods.

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- X is *crooked at* $\langle A, B, U, V \rangle$ if there are closed sets $F_0, F_1, F_2 \subseteq X$ such that $A \subseteq F_0$, $B \subseteq F_2$, $F_0 \cup F_1 \cup F_2 = X$, $F_0 \cap F_1 \subseteq V$, $F_1 \cap F_2 \subseteq U$, $F_0 \cap F_2 = \emptyset$,

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Theorem [Krasinkiewicz–Minc, 1977]

A compact Hausdorff space X is hereditarily indecomposable if and only if it is crooked.

Crookedness – maps

Definition [Maćkowiak, 1985]

Let $f\colon X \to Y$ be a continuous map, $\langle A, B, U, V \rangle$ admissible in Y.

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Definition

Let $f \colon X \to \langle Y, d \rangle$ be a continuous map, $A, B \subseteq Y$ closed disjoint, and $\varepsilon > 0$.

- f is *ε-crooked at* $\langle A, B \rangle$ if it is crooked at $\langle A, B, A^\varepsilon, B^\varepsilon \rangle$.
- f is *ε-crooked* if it is ε-crooked at every closed disjoint $\langle A, B \rangle$.

Crookedness

Proposition

A continuous map $f\colon \mathbb{I} \to \langle X, d\rangle$ is ε-crooked if and only if it satisfies the classical definition: for every $x \leq y \in \mathbb{I}$ there are $x \leq y' \leq x' \leq y$ such that $f(x) \approx_\varepsilon f(x')$ and $f(y) \approx_\varepsilon f(y')$.

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For Peano continua, this was essentially proved by [Brown, 1960].

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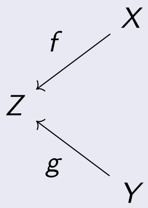

Fraïssé theory – almost amalgamation property

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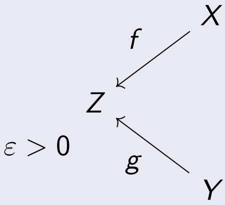

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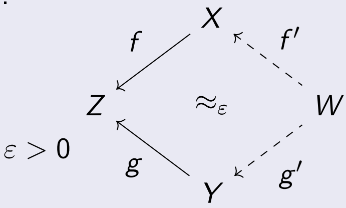

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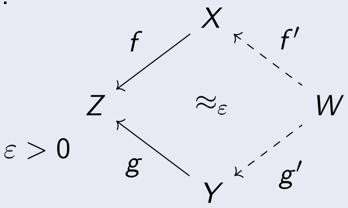

Fact

The interval category $\mathcal{I}$ has the almost amalgamation property by the mountain climbing theorem.

Fraïssé theory – sequences

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$$Y$$

Fraïssé theory – sequences

Definition

A sequence $\langle X_*, f_* \rangle$ in a category of compacta $\mathcal{K}$ is

- *universal* if for every $\mathcal{K}$-object Y there is a $\mathcal{K}$-map $g \colon X_n \to Y$;

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \cdots \longleftarrow X_n \xleftarrow{f_n} \cdots$$

$$X_n \overset{g}{\dashrightarrow} Y$$

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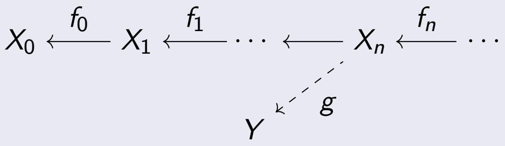

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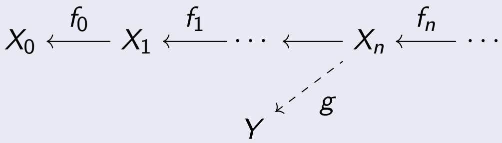

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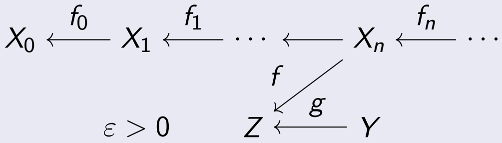

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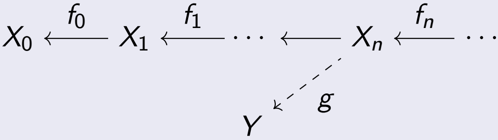

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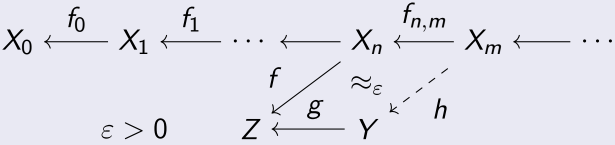

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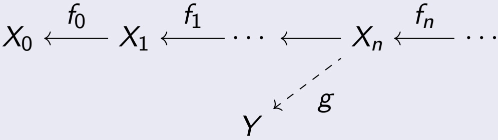

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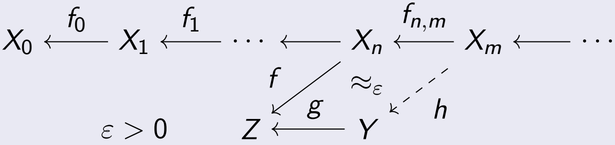

- *Fraïssé* if it is both universal and almost projective.

Fraïssé theory

Theorem

A category of compacta $\mathcal{K}$ has a Fraïssé sequence if and only if it

1. is *directed*,
2. has a countable *universal family of objects*,
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Hence, $\mathcal{I}$ has a Fraïssé sequence.

Definition

Let $\mathcal{K}$ be category of compacta and let X be a compactum. The *Banach–Mazur game* $\mathrm{BM}_{\mathcal{K}}(X)$ is defined as follows. Eve starts with a $\mathcal{K}$-map $f_0\colon X_0 \leftarrow X_1$, Odd responds with a $\mathcal{K}$-map $f_1\colon X_1 \leftarrow X_2$, and so on. The outcome of the play is the sequence $\langle X_*, f_* \rangle$, and Odd wins if $X_\infty \cong X$. The space X is *generic over* $\mathcal{K}$ if Odd has a winning strategy for $\mathrm{BM}_{\mathcal{K}}(X)$.

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Observation

The generic object over $\mathcal{K}$ is unique (if it exists).

Fraïssé theory – games and uniqueness

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Theorem

The limit of a Fraïssé sequence in $\mathcal{K}$ is generic over $\mathcal{K}$. Therefore, the *Fraïssé limit* is unique.

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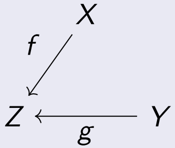

Fraïssé theory – large objects

Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. An $\mathcal{L}$-object X is

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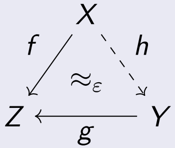

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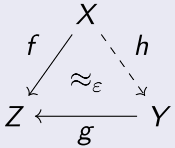

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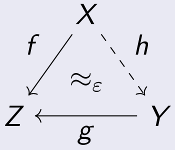

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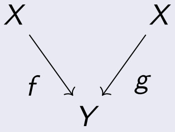

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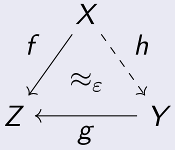

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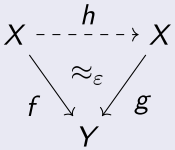

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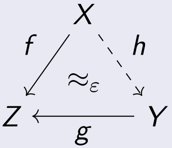

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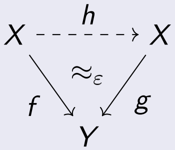

We say just "in $\mathcal{L}$" instead of "in $\langle \mathcal{L}, \mathcal{L} \rangle$".

Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories of compacta. We consider the following condition for $\langle \mathcal{K}, \mathcal{L} \rangle$.

Fraïssé theory – condition (F)

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(F) For every sequence $\langle X_*, f_* \rangle$ in $\mathcal{K}$, every $\mathcal{L}$-map $g \colon X_\infty \to Y$ to a $\mathcal{K}$-object Y, and every $\varepsilon > 0$

$$\begin{array}{ccccccccc} X_0 & \xleftarrow{f_0} & X_1 & \xleftarrow{f_1} & X_2 & \xleftarrow{f_2} & X_3 & \xleftarrow{f_3} \cdots & X_\infty \\ & & & & & & & \swarrow g & \\ & & & & & & Y & & \end{array}$$

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Fact

It follows from the result [Mardešić–Segal, 1963] that $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$ satisfies (F).

Fraïssé theory

Theorem

Let $\mathcal{K}$ be a category of compacta such that all $\mathcal{K}$-maps are surjections, and $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ satisfies (F). Then the following conditions are equivalent.

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Hence, there is a unique Fraïssé limit in $\sigma\mathcal{I}$ satisfying all the conditions.

The pseudo-arc

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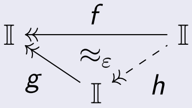
- By the result of Bing, the Fraïssé limit of $\sigma\mathcal{I}$ is the pseudo-arc.

The pseudo-arc

Theorem

For every $\mathcal{I}$-map g and every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-crooked $\mathcal{I}$-map f there is an $\mathcal{I}$-map h such that $f \approx_\varepsilon g \circ h$.



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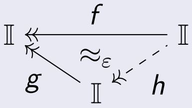

Corollary

- The almost amalgamation property of $\mathcal{I}$ follows from the fact that there is an ε-crooked $\mathcal{I}$-map for every $\varepsilon > 0$.

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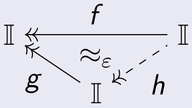

Corollary

- The almost amalgamation property of $\mathcal{I}$ follows from the fact that there is an ε-crooked $\mathcal{I}$-map for every $\varepsilon > 0$.
- Every crooked sequence in $\mathcal{I}$ is almost projective, and hence Fraïssé. Therefore, there is a unique hereditarily indecomposable arc-like continuum.

Conclusion

Theorem

Let $\langle X_*, f_* \rangle$ be a sequence in $\mathcal{I}$. The following conditions are equivalent.

1. X_∞ is hereditarily indecomposable.
2. X_∞ is crooked.
3. The maps $f_{n,\infty}$ are crooked.
4. $\langle X_*, f_* \rangle$ is a crooked sequence.
5. $\langle X_*, f_* \rangle$ is a Fraïssé sequence.
6. X_∞ is universal and almost projective in $\sigma\mathcal{I}$.
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Thank you for your attention.