Topological Games of Bounded Selections

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Definition

Let $\mathcal{A}, \mathcal{B}$ be families of sets and $k \in \mathbb{N}$. We denote by $\mathrm{G_k}(\mathcal{A}, \mathcal{B})$ the following game played between ALICE and BOB.

- In each inning $n \in \omega$ ALICE chooses $A_n \in \mathcal{A}$ and BOB responds with $B_n \subset A_n$ such that $|B_n| \leq k$.
- We then say that BOB wins if $\bigcup_{n\in\omega} B_n \in \mathcal{B}$ and that ALICE wins otherwise.

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Let $\mathcal{A}, \mathcal{B}$ be families of sets. We denote by $\mathrm{G_{fin}}(\mathcal{A}, \mathcal{B})$ the following game played between ALICE and BOB.

- In each inning $n \in \omega$ ALICE chooses $A_n \in \mathcal{A}$ and BOB responds with $B_n \subset A_n$ finite.
- We then say that BOB wins if $\bigcup_{n \in \omega} B_n \in \mathcal{B}$ and that ALICE wins otherwise.

Example

Given a space (X, τ), let

$$\mathcal{A} = \mathcal{B} = \mathcal{O} = \left\{ \mathcal{U} \subset \tau : X = \bigcup \mathcal{U} \right\} \text{(covering games)}$$

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Example

Given a space (X, τ) and $p \in X$, let

$$\mathcal{A} = \mathcal{B} = \Omega_p = \left\{ A \subset X : p \in \overline{A} \right\} \text{(tightness games)}$$

Definition

A **strategy** for PLAYER is a function whose input is the history of the game up to a given PLAYER's turn and output is a valid response of PLAYER to that history.

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Definition

Two games G_1 and G_2 are **equivalent** if

$$\text{ALICE} \uparrow \mathsf{G}_1 \iff \text{ALICE} \uparrow \mathsf{G}_2$$
$$\text{BOB} \uparrow \mathsf{G}_1 \iff \text{BOB} \uparrow \mathsf{G}_2$$

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Definition

Let $\mathcal{A}, \mathcal{B}$ be families of sets. We denote by $\mathsf{G}_{\mathrm{bnd}}(\mathcal{A}, \mathcal{B})$ the following game played between ALICE and BOB. In each inning $n \in \omega$ ALICE chooses $A_n \in \mathcal{A}$ and BOB responds with $B_n \subset A_n$ finite. We then say that BOB wins if:

- There is a $k \in \mathbb{N}$ such that $|B_n| \leq k$ for every $n \in \omega$;
- $\bigcup_{n \in \omega} B_n \in \mathcal{B}$,

and that ALICE wins otherwise.

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Theorem

$\text{ALICE} \uparrow \mathsf{G}_{\mathrm{bnd}}(\Omega_p, \Omega_p)$ *if, and only if,* $\text{ALICE} \uparrow \mathsf{G}_{\mathrm{k}}(\Omega_p, \Omega_p)$ *for every* $k \in \mathbb{N}$.

Theorem

$\text{ALICE} \uparrow \mathsf{G}_{\mathrm{bnd}}(\Omega_p, \Omega_p)$ if, and only if, $\text{ALICE} \uparrow \mathsf{G}_{\mathrm{k}}(\Omega_p, \Omega_p)$ for every $k \in \mathbb{N}$.

Therefore, $\mathsf{G}_{\mathrm{bnd}}(\Omega_p, \Omega_p)$ is **not** equivalent to $\mathsf{G}_{\mathrm{fin}}(\Omega_p, \Omega_p)$:

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Proposition

Over $C_p(\mathbb{R})$:

(a) $\text{BOB} \uparrow \mathsf{G}_{\mathrm{fin}}(\Omega_0, \Omega_0)$ *(D. Barman, A. Dow (2011));*

(b) $\text{ALICE} \uparrow \mathsf{G}_{\mathrm{k}}(\Omega_0, \Omega_0)$ *for every $k \in \mathbb{N}$ (M. Sakai (1988)).*

Theorem

$\textsc{Bob} \uparrow \mathsf{G}_{\mathrm{bnd}}(\Omega_p, \Omega_p)$ *if, and only if, there is an* $m \in \mathbb{N}$ *such that* $\textsc{Bob} \uparrow \mathsf{G}_{\mathrm{m}}(\Omega_p, \Omega_p)$.

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Therefore, $\mathsf{G}_{\text{bnd}}(\Omega_p, \Omega_p)$ is also **not** equivalent to $\mathsf{G}_{\text{k}}(\Omega_p, \Omega_p)$ for any $k \in \mathbb{N}$:

Theorem

$\text{Bob}\uparrow \mathsf{G}_{\mathrm{bnd}}(\Omega_p, \Omega_p)$ if, and only if, there is an $m \in \mathbb{N}$ such that $\text{Bob}\uparrow \mathsf{G}_{\mathrm{m}}(\Omega_p, \Omega_p)$.

Therefore, $\mathsf{G}_{\mathrm{bnd}}(\Omega_p, \Omega_p)$ is also **not** equivalent to $\mathsf{G}_{\mathrm{k}}(\Omega_p, \Omega_p)$ for any $k \in \mathbb{N}$:

Proposition (L. Aurichi, A. Bella, R. Dias (2018))

For each $k \in \mathbb{N}$ there is a countable space X_k with only one non-isolated point p_k on which $\text{Alice}\uparrow \mathsf{G}_{\mathrm{k}}(\Omega_{p_k}, \Omega_{p_k})$ and $\text{Bob}\uparrow \mathsf{G}_{\mathrm{k+1}}(\Omega_{p_k}, \Omega_{p_k})$.

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Proposition

$\text{Bob} \uparrow \mathsf{G}_{\text{bnd}}(\mathcal{O}, \mathcal{O})$ *over every compact space, but*
$\text{Alice} \uparrow \mathsf{G}_{\text{k}}(\mathcal{O}, \mathcal{O})$ *over* 2^ω *for every* $k \in \mathbb{N}$.
Moreover, $\text{Bob} \uparrow \mathsf{G}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ *over every* σ*-compact space, but*
$\text{Alice} \uparrow \mathsf{G}_{\text{bnd}}(\mathcal{O}, \mathcal{O})$ *over* $\mathbb{R}$.

The following game is very useful to understand $\mathrm{G}_{\mathrm{bnd}}(\mathcal{O}, \mathcal{O})$:

Definition

Let $k \in \mathbb{N}$, and $\mathcal{A}, \mathcal{B}$ be families of sets. We denote by $\mathrm{G}_{\mathrm{k}}(\mathcal{A}, \mathcal{B}) \bmod 1$ the following game, played between ALICE and BOB.

1. in the first inning, ALICE chooses $A_0 \in \mathcal{A}$ and BOB responds with $B_0 \subset A_0$ finite;
2. in every inning $n \in \omega$ after that, ALICE chooses $A_n \in \mathcal{A}$ and BOB responds with $B_n \subset A_n$ such that $|B_n| \leq k$.

We then say that BOB wins if $\bigcup_{n \in \omega} B_n \in \mathcal{B}$ and that ALICE wins otherwise.

Theorem

Over every space X,

$$\text{ALICE} \uparrow \mathsf{G}_{\text{bnd}}(\mathcal{O}, \mathcal{O}) \iff \text{ALICE} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O}) \bmod 1.$$

Moreover, if X is a Hausdorff space, then

$$\text{BOB} \uparrow \mathsf{G}_{\text{bnd}}(\mathcal{O}, \mathcal{O}) \iff \text{BOB} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O}) \bmod 1.$$

Proof uses

Theorem (L. Crone, L. Fishman, N. Hiers and S. Jackson (2018))

Let X be a space and $k \in \mathbb{N}$. Then

$$\text{ALICE} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O}) \iff \text{ALICE} \uparrow \mathsf{G}_k(\mathcal{O}, \mathcal{O}).$$

Moreover, if X is a Hausdorff space, then

$$\text{BOB} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O}) \iff \text{BOB} \uparrow \mathsf{G}_k(\mathcal{O}, \mathcal{O}).$$

Theorem

Let X be a regular space. Then $\text{Bob} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O}) \bmod 1$ *if, and only if, there is a compact set $K \subset X$ such that, for every open set $V \supset K$,* $\text{Bob} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O})$ *over $X \setminus V$.*

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Corollary

Let X be a regular space. Then $\text{Bob} \uparrow \mathsf{G}_{\text{bnd}}(\mathcal{O}, \mathcal{O})$ *if, and only if, there is a compact set $K \subset X$ such that, for every open set $V \supset K$,* $\text{Bob} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O})$ *over $X \setminus V$.*

And we can characterize even stricter subsets of metrizable spaces:

Theorem (R. Telgársky (1975), F. Galvin (1978))

Let X be a space in which every point is a G_δ set. Then $\textsc{Bob} \uparrow \mathsf{G}_1(\mathcal{O}, \mathcal{O})$ *if, and only if, X is countable.*

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Corollary

Let X be a regular space in which every compact set is a G_δ set (e.g., a metrizable space). Then $\text{Bob} \uparrow \mathsf{G}_{\mathrm{bnd}}(\mathcal{O}, \mathcal{O})$ if, and only if, there is a compact set $K \subset X$ and a countable set $N \subset X$ such that $X = K \cup N$.

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Děkuji!