Products of $\gamma$-sets

Magdalena Włodecka

Cardinal Stefan Wyszyński University in Warsaw, Poland

Winter School 2019

(joint work with P. Szewczak)
The space of continuous functions

\( X = \) a subset of \( \mathbb{R} \)

\( C(X) := \{ f : X \to \mathbb{R} : f \text{ is continuous} \}, \text{w.r.t pointwise conv. topology} \)
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A space $E$ is a Fréchet-Urysohn space if, for every point $q \in E$ and every subset $A \subseteq E$ such that $q \in \overline{A}$, there is $\{q_n : n \in \mathbb{N}\} \subseteq A$ such that $\lim_{n \to \infty} q_n = q$.

$q \in \overline{A} \iff A \ni q_n \longrightarrow q$
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C(X) \text{ is Fréchet-Urysohn} & \iff X \text{ is ???}
\end{align*}
**γ-sets**

An infinite open cover $\mathcal{U}$ of a space $X$ such that $X \notin \mathcal{U}$ is:

- **ω-cover**, if for every finite $F \subseteq X$, there is $U \in \mathcal{U}$ such that $F \subseteq U$,

- **γ-cover**, if for every $x \in X$, a set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

**γ-set** = every ω-cover has a γ-subcover.

**Theorem 1 (Gerlits–Nagy)**

$C(X)$ is Fréchet-Urysohn $\iff X$ is γ.

Is there an uncountable γ-set?
γ-sets

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Is there an uncountable $\gamma$-set?
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$p$ – the minimal cardinality of a centered family in $[\mathbb{N}]^\infty$ with no pseudointersection
\( \gamma \)-sets

\( \wp \) – the minimal cardinality of a centered family in \([\mathbb{N}]^\infty\) with no pseudointersection

**Theorem 2 (Recław)**

\( X \) is \( \gamma \) \( \iff \) \( \forall \varphi : X \xrightarrow{\text{cont.}} [\mathbb{N}]^\infty \) with a centered image, \( \varphi[X] \) has a pseudointersection
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**Corollary 3**

- \(|X| < \rho \iff X \text{ is } \gamma\)
- There is \(X \subseteq [\mathbb{N}]^\infty\), of cardinality \(\rho\), which is not \(\gamma\)
**$\gamma$-sets**

$p$ – the minimal cardinality of a centered family in $[\mathbb{N}]^\infty$ with no pseudointersection

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**Corollary 4**

\( \omega_1 < p \implies \text{there exists an uncountable } \gamma\text{-set} \)

Is there a \( \gamma\text{-set of cardinality } \geq p \)?
Tower in $[\mathbb{N}]^\infty$

$[\mathbb{N}]^\infty \supseteq T = \{x_\alpha : \alpha < \kappa\}$ is a tower, if $x_\beta \subseteq^* x_\alpha$ for $\alpha < \beta$
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$\omega < p \leq b \leq c$
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**Lemma 5**

\[ T \text{ exists} \iff p = b \]
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**Theorem 6 (Tsaban)**

\[ T \cup \text{Fin} \text{ is a } \gamma\text{-set} \]
Products of $\gamma$-sets

Theorem 7 (Miller, Tsaban, Zdomskyy)

Assuming CH, there are $\gamma$-sets $X$ and $Y$ such that $X \times Y$ is not Menger space.
## Products of $\gamma$-sets

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**Corollary 9 (Szewczak, MW)**

$$(T \cup \text{Fin}) \times (\tilde{T} \cup \text{Fin}) \text{ is } \gamma$$
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\[ \kappa := \min \{|X| : X \text{ is not productively } \gamma \} \]
Products of $\gamma$-sets

$$\kappa := \min\{|X| : X \text{ is not productively } \gamma\}$$

**Theorem 10 (Szewczak, MW)**

Let $\kappa = b$ and $Y \subseteq P(\mathbb{N})$ be a $\gamma$-set. Then $(T \cup \text{Fin}) \cup Y$ is $\gamma$.

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**Theorem 12 (Szewczak, MW)**

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