Products of γ-sets

Magdalena Włudecka

Cardinal Stefan Wyszyński University in Warsaw, Poland

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(joint work with P. Szewczak)

The space of continuous functions

$X =$ a subset of $\mathbb{R}$

$\mathrm{C}(X) := \{f : X \to \mathbb{R} : f$ is continuous$\}$,w.r.t pointwise conv. topology

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A space E is a Fréchet-Urysohn space if, for every point $q \in E$ and every subset $A \subseteq E$ such that $q \in \overline{A}$, there is $\{q_n : n \in \mathbb{N}\} \subseteq A$ such that $\lim_{n\to\infty} q_n = q$.

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γ-sets

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Is there an uncountable γ-set?

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X is $\gamma \iff \forall\, \varphi : X \xrightarrow{cont.} [\mathbb{N}]^\infty$ with a centered image, $\varphi[X]$ has a pseudointersection

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- $|X| < \mathfrak{p} \Longrightarrow X$ *is* γ
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Is there a γ-set of cardinality $\geqslant \mathfrak{p}$?

Tower in $[\mathbb{N}]^\infty$

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$$\omega < \mathfrak{p} \leqslant \mathfrak{b} \leqslant \mathfrak{c}$$

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Lemma 5

T exists $\iff \mathfrak{p} = \mathfrak{b}$

Theorem 6 (Tsaban)

$T \cup \mathrm{Fin}$ is a γ-set

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Assuming CH, there are γ-sets X and Y such that $X \times Y$ is not Menger space.

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Corollary 9 (Szewczak, MW)

$(T \cup \mathrm{Fin}) \times (\widetilde{T} \cup \mathrm{Fin})$ *is* γ

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Let $\kappa = \mathfrak{b}$ *and* $Y \subseteq \mathrm{P}(\mathbb{N})$ *be a* γ*-set. Then* $(T \cup \mathrm{Fin}) \sqcup Y$ *is* γ*.*

Corollary 11 (Szewczak, MW)

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Theorem 12 (Szewczak, MW)

Let $\lambda < \mathfrak{b}$. *Then* $\bigsqcup_{\beta<\lambda}(T_\beta \cup \mathrm{Fin})$ *is countably* γ.

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Let $\omega_1 < \mathfrak{b}$. Then $X = \bigsqcup_{\beta<\omega_1}(T_\beta \cup \mathrm{Fin})$ is countably γ, X is not γ, $|X| = \mathfrak{p}$ and X is a metrizable space.

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Corollary 14 (Szewczak, MW)

Let $\lambda < \mathfrak{b}$. Then $\bigcup_{\beta<\lambda}(T_\beta \cup \mathrm{Fin})$ is γ.