Uniformization properties of ladder systems under $MA(S)[S]$

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Overview

1. Definitions
2. Forcing with a Souslin tree
3. $MA(S)[S]$
4. Anti-uniformization properties

Basic definitions

Definition

A ladder system over a stationary subset $E \subseteq \lim(\omega_1)$ is a sequence $L = \langle L_\alpha : \alpha \in E \rangle$ such that $ot(L_\alpha) = \omega$ and $\bigcup L_\alpha = \alpha$ for every $\alpha \in E$.

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Definition

A ladder system L is uniformizable if for all sequence of colourings $\langle f_\alpha : L_\alpha \to \omega | \alpha \in E \rangle$, there is a single function $f : \omega_1 \to \omega$ which almost equals all f_α (i.e., $f \restriction L_\alpha =^* f_\alpha$).

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The space X_L

Given a ladder system L, we define the space $X_L = \omega_1 \times \{0\} \cup E \times \{1\}$, where the points of $\omega_1 \times \{0\}$ are isolated and a base neighborhood at $(\alpha, 1) \in E \times \{1\}$ is of the form $(\alpha, 1) \cup (L_\alpha \setminus F \times \{0\})$ where F is finite

Weak uniformizations

Definition

A ladder system is said to satisfy $\mathcal{M}_n$ if for each $f : E \to \omega$ there is a function $F : \omega_1 \to [\omega]^{n+1}$ such that $f(\alpha) \in F(L_\alpha(n))$ for all $\alpha \in E$ and all but finitely many $n \in \omega$.

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A ladder system is said to satisfy $\mathcal{M}_{<\omega}$ if for each $f : E \to \omega$ there is a function $F : \omega_1 \to [\omega]^{<\omega}$ such that $f(\alpha) \in F(L_\alpha(n))$ for all $\alpha \in E$ and all but finitely many $n \in \omega$.

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Topological equivalences

- X_L is normal iff L satisfies $\mathcal{P}_0$.
- X_L is countably metacompact iff L satisfies $\mathcal{M}_{<\omega}$.
- If L satisfies $\mathcal{P}_{<\omega}$, then X_L is countably paracompact.

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It is still open if the converse of the last point holds.

Anti-uniformization properties

Definition

A ladder system satisfies G_i if for each $f : \omega_1 \to \omega$ the set $\{\alpha \in E : \varphi_i(f, \alpha)\}$ is nonstationary

- $\varphi_1(f, \alpha) \equiv |f''(L_\alpha)| = \aleph_0$
- $\varphi_2(f, \alpha) \equiv f \restriction L_\alpha$ is finite to one
- $\varphi_3(f, \alpha) \equiv f \restriction L_\alpha$ is eventually one to one

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Observations

- A $\clubsuit$-sequence satisfies H_1
- A ladder system L satisfies H_2 iff $E \times \{1\}$ is not a G_δ set in X_L.

Some results

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Theorem (Shelah)

It is consistent that there exists a ladder system L which satisfies $\mathcal{P}_0$ and H_2.

Forcing with a Souslin tree

Lemma

Let $L = \langle L_\alpha : \alpha \in E \rangle$ be a ladder system. The space X_L is countably paracompact iff for all $f : E \to \omega$, there exists $F : \omega_1 \to [\omega]^{<\omega}$ and $g : E \to [\omega]^{<\omega}$ such that

$$f(\alpha) \in F(\beta) \subseteq g(\alpha)$$

for all $\alpha \in E$ and for all but finitely many $\beta \in L_\alpha$.

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Theorem

After forcing with a Souslin tree S the following hold for every ladder system L:

- *X_L is not countably paracompact*
- *L does not satisfy $\mathcal{M}_n$ for every $n \in \omega$*

Proof

Sketch of proof

- Assume $S \subseteq \omega^{<\omega_1}$ is a Souslin tree and let $b \subseteq S$ be a generic branch. Also, let $\dot{L} = \langle \dot{L}_\alpha : \alpha \in \dot{E} \rangle$ be a name for a ladder system.

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- Find a club $C \subseteq \omega_1$ such that for every $s \in S$ with $l(s) = \alpha^+$, s decides "$\alpha \in \dot{E}$" and $\dot{L}_\alpha$, where $\alpha^+ = \min\{\beta \in C : \beta > \alpha\}$.

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- In $V[b]$ define $f : \omega_1 \to \omega$ such that $f(\alpha) = b(\alpha^+)$.

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- In $V[b]$ define $f : \omega_1 \to \omega$ such that $f(\alpha) = b(\alpha^+)$.
- Let $t \in S$ and let $\dot{F}$ be a name for a function from ω_1 to $[\omega]^{<\omega}$. Using elementary submodels, we can find a level $\delta \in \omega_1$ (with $\delta > l(t)$) and $s \geq t$ such that $l(s) = \delta^+$, $s \Vdash$"$\delta \in \dot{E}$" and s decides $\dot{F} \restriction L_\delta$.

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- Find a club $C \subseteq \omega_1$ such that for every $s \in S$ with $l(s) = \alpha^+$, s decides "$\alpha \in \dot{E}$" and $\dot{L}_\alpha$, where $\alpha^+ = \min\{\beta \in C : \beta > \alpha\}$.
- In $V[b]$ define $f : \omega_1 \to \omega$ such that $f(\alpha) = b(\alpha^+)$.
- Let $t \in S$ and let $\dot{F}$ be a name for a function from ω_1 to $[\omega]^{<\omega}$. Using elementary submodels, we can find a level $\delta \in \omega_1$ (with $\delta > l(t)$) and $s \geq t$ such that $l(s) = \delta^+$, $s \Vdash$"$\delta \in \dot{E}$" and s decides $\dot{F} \restriction L_\delta$.
- Finally, if the set $H = \bigcap_{n\in\omega} \bigcup_{m\geq n} F(L_\delta(m))$ is infinite, then no g can satisfy the conclusion of the lemma. On the other hand, if H is finite, we are free to put the value of $f(\delta)$ out of this set.

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- Thus, a model obtained by a forcing extension with a Souslin tree is the opposite of a model of *MA* also for uniformization properties.
- What happens in models where *MA* and Souslin trees are combined? Is there any uniformization property for ladder systems?

Definitions

Definition (Larson, Todorcevic)

$MA(S)$ is the assertion that there exists a (coherent) Souslin tree S such that for every poset $\mathbb{P}$ which satisfies that $\mathbb{P} \times S$ is ccc and for every family $\mathcal{D} = \{D_\alpha : \alpha \in \omega_1\}$ of dense subsets of $\mathbb{P}$, there exists a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$.

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Notation

$MA(S)[S]$ implies $\varphi \Leftrightarrow \varphi$ is true in any model obtained by a forcing extension with the Souslin tree S over a model of $MA(S)$.

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Observation

A total ladder system $L = \langle L_\alpha : \alpha \in E \rangle$ is a ladder system in which $E = \lim(omega_1)$. Note that the property $\mathcal{M}_{<\omega}$ is hereditary respect to the stationary set.

$MA(S)[S]$

Lemma

Let $B = \{t_\alpha : \alpha < \omega_1\} \subseteq S$ be uncountable. Then for every $\gamma \in \omega_1$, there exists a chain $\{t_{\alpha_\xi} : \xi \in \gamma\} \subseteq B$ with order type γ.

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Theorem

$MA(S)[S]$ implies that all (total) ladder system satisfy $\mathcal{M}_{<\omega}$.

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- Let V be a model of $MA(S)$. Let $\dot{L} = \langle \dot{L}_\alpha : \alpha \in \lim(\omega_1)\rangle$ be an S-name for a total ladder system and let $\dot{f}$ be an S-name for a function from $\lim(\omega_1)$ to ω.

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- Assume WLOG that for every $\alpha \in \omega_1$ and every $s \in S$ such that $l(s) = \alpha + 1$, s decides $\dot{f}(\alpha)$ and $\dot{L}_\alpha$.

Proof

Sketch of proof

- Define the forcing $\mathbb{P} = \mathbb{P}(\dot{f}, \dot{L})$ as follows:

$$\mathbb{P} = \{(p, F) : p \in Fin(S, [\omega]^{<\omega}) \wedge F \in [\lim(\omega_1)]^{<\omega}\}$$

and $(p, F) \leq (q, G)$ iff $p \supseteq q$, $F \supseteq G$ and
$\forall s \in dom(p) \setminus dom(q)\ \forall \alpha \in G\ \forall t \in A(p)$

$$\Big((s \subseteq t) \wedge (l(t) > \alpha) \wedge (t \Vdash \text{“}l(s) \in \dot{L}_\alpha \wedge \dot{f}(\alpha) = n\text{''})\Big) \implies (n \in p(s))$$

where $A(p)$ is the set of maximal elements of domain of p.

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- Given a generic branch $b \subseteq S$, $h_G \restriction b$ looks like a function from ω_1 to $[\omega]^{<\omega}$ which uniformizes f in the sense of $\mathcal{M}_{<\omega}$.

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- In order to prove the existence of such generic filter, prove that $\mathbb{P} \times S$ is ccc.
- For this, let $\langle((p_\alpha, F_\alpha), t_\alpha) : \alpha \in \omega_1\rangle \subseteq \mathbb{P} \times S$ and assume $\{dom(p_\alpha) : \alpha \in \omega_1\}$ and $\{F_\alpha : \alpha \in \omega_1\}$ form Δ-systems and that each element in these sets are "far enought" from each other (with exception of the roots).

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- Using the previous lemma, we can also assume that $\{t_\alpha : \alpha \leq \omega\}$ is an S-chain.

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- Using the previous lemma, we can also assume that $\{t_\alpha : \alpha \leq \omega\}$ is an S-chain.
- Finally, find $n \in \omega$ such that (p_ω, F_ω) and (p_n, F_n) are compatible. Thus since t_ω and t_n are in the chain, $((p_\omega, F_\omega), t_\omega)$ and $((p_n, F_n), t_n)$ are compatible as well.

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A ladder system satisfies H_3 if for each $f : \omega_1 \to \omega$ the set $\{\alpha \in E : f \restriction L_\alpha$ is not eventually one-to-one$\}$ is stationary.

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Note that we just have to prove that for every total ladder system $L = \langle L_\alpha : \alpha \in \lim(\omega_1) \rangle$ there exists a function $f : \omega_1 \to \omega$ such that $f \upharpoonright L_\alpha$ is eventually one-to-one for every $\alpha \in \lim(\omega_1)$.

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Since S doesn't add reals and is ccc, if $L = \langle L_\alpha : \alpha \in \lim(\omega_1) \rangle$ is a total ladder system in the extension, there exists a set $L' = \{L^n_\alpha : \alpha \in \lim(\omega_1) \wedge n \in \omega\}$ in the ground model such that $L_\alpha \in \{L^n_\alpha : n \in \omega\}$ for each α and in consequence it is suffices to prove that the following holds:

Theorem

For every family $L = \{L_\alpha^n : \alpha \in \lim(\omega_1) \wedge n \in \omega\}$ *(where each* L_α^n *is a ladder in* α*) there exists a function* $f : \omega_1 \to \omega$ *such that* $f \restriction L_\alpha^n$ *is eventually one-to-one for every* $\alpha \in \lim(\omega_1)$ *and every* $n \in \omega$*.*

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Let $\mathbb{P} = \mathbb{P}(L) = \{(p, F) : p \in Fn(\omega_1, \omega) \wedge F \in [\omega_1 \times \omega]^{<\omega}\}$ and let $(p, F) \leq (q, G)$ iff $p \supseteq q, F \supseteq G$ and $(p \setminus q) \restriction L_\alpha^n$ is one-to-one for every $(\alpha, n) \in G$.

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Repeat the scheme of the last theorem using this poset.

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Question

Is there (consistently) a ladder system which satisfies $\mathcal{M}_{<\omega}$ and G_1?

Thank you!