Borel ideals and Ramsey spaces

Sonia Navarro Flores

National University of Mexico
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Ideals

Definition

A family $\mathcal{I} \subset \mathcal{P}(X)$ of subsets of a given set X is an *ideal* on X if

1. for $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$,
2. for $A, B \subset X$, $A \subset B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$ and
3. $X \notin \mathcal{I}$.

A filter is the dual notion of ideal.

Examples

1. The eventually different ideal $\mathcal{ED}$ is such that $A \in \mathcal{ED}$ iff $(\exists m, n \in \omega)(\forall k > n)(|\{l : (k, l) \in A\}| \leq m)$
2. $\mathcal{ED}_{\mathrm{fin}} = \{A \cap \Delta : A \in \mathcal{ED}\}$ where $\Delta = \{(n, m) \in \omega \times \omega : n \leq m\}$.
3. The ideal $\mathrm{Fin} \times \mathrm{Fin}$ is such that $A \in \mathrm{Fin} \times \mathrm{Fin}$ iff $\{n : \{m : (n, m) \in A\} \notin \mathrm{Fin}\} \in \mathrm{Fin}$.
4. The summable ideal $\mathcal{I}_{\frac{1}{n}} = \{A \subseteq \omega : \Sigma_{n \in A} \frac{1}{n} < \infty\}$.

Definition

Given an ideal $\mathcal{I}$ on X:

1. We denote by $\mathcal{I}^+$ the family of $\mathcal{I}$-positive sets, i.e. subsets of X which are not in $\mathcal{I}$.
2. If $Y \in \mathcal{I}^+$, we denote by $\mathcal{I} \upharpoonright Y$ the ideal $\{I \cap Y : I \in \mathcal{I}\}$ on Y.

We will consider ideals on countable sets, so we pretend that they are, in fact, ideals on ω.

This order, called *Katětov* order was introduced by M. *Katětov* in 1968 to study convergence in topological spaces. It has been used to clasify ultrafilters and ideals for mathematicians as Baumgartner, Solecky, Brendle and Hrusak.

Definition

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ on ω we shall say that:

1. $\mathcal{I}$ is *Katětov* below $\mathcal{J}$ ($\mathcal{I} \leq_K \mathcal{J}$) if there is a function $f : \omega \to \omega$ such that for all $I \in \mathcal{I}$, $f^{-1}[I] \in \mathcal{J}$.
2. $\mathcal{I}$ and $\mathcal{J}$ are *Katětov* equivalent ($\mathcal{I} \simeq_K \mathcal{J}$) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$.

Rudin-Keisler order

Definition

Let $\mathcal{F}$, $\mathcal{G}$ be filters on ω, we say that $\mathcal{F}$ is Rudin-Keisler below $\mathcal{G}$ ($\mathcal{F} \leq_{\mathrm{RK}} \mathcal{G}$) if there exists a function $f : \omega \to \omega$ such that $X \in \mathcal{F}$ if and only if $f^{-1}[X] \in \mathcal{G}$.

Topological Ramsey spaces are an abstraction of the Ellentuck space that satisfy a general version of Ellentuck's Theorem. In the book "Introduction to Ramsey spaces", Todorcevic defines axioms $\mathbf{A}.\mathbf{1}-\mathbf{A}.\mathbf{4}$ by extracting properties of the Ellentuck space building on prior work of Carlson-Simpson. Topological Ramsey spaces has been studied by several mathematicians because of their applications to Tukey order theory and Bannach spaces.

If $T \subset [\omega]^{<\omega}$ and $\sqsubset$ is an end-extension order on T, we say that T is a tree if for every $t \in T$, the set $\{s \in T : s \sqsubset t\}$ is well ordered. If $(T, \sqsubset)$ is a tree,
$[T] = \{\bigcup C \subseteq \omega : C \in [T]^{\omega} \text{and} (\forall s, t \in C)(s \sqsubset t$ or $t \sqsubset s\}$.

Definition.

An ideal $\mathcal{I}$ is a TRS if there exists some tree $T \subset [\omega]^{<\omega}$ such that:

- For every $t \in T$, there is some $s \in T$ such that $t \sqsubset s$.
- For every $t \in T$, $X \in [T]$ such that $t \subset X$, there exists $Y \in [T]$ such that $t \sqsubset Y$ and $Y \subseteq X$.
- For every $t \in T$, $X \in [T]$ such that $t \subset X$, if $\mathcal{O} \subset succ(t)$, there exists $Y \in [T]$ such that $Y \subseteq X$ and $succ(t) \cap [Y]^{<\omega} \subseteq \mathcal{O}$ or $succ(t) \cap [Y]^{<\omega} \subseteq \mathcal{O}^c$.
- For every $X \in \mathcal{I}^+$ there exists $Y \in [T]$ such that $Y \subseteq X$.

If $\mathcal{I}$ is an ideal TRS with witness T, $([T], \subseteq)$ is a topological Ramsey space.

Nash-Williams Theorem (Todorcevic)

For every family $\mathcal{F}$ of $\sqsubset$ $-$incompatible members of T, if $X \in [T]$ and $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ then there exist $i_0 \in 2$ and $Y \in [T] \cap [X]^\omega$ such that $[Y]^{<\omega} \cap \mathcal{F}_{i_0} = \emptyset$.

Let $\mathcal{I}_{\mathcal{H}^2}$ be the ideal such that $A \notin \mathcal{I}_{\mathcal{H}^2}$ iff $(\forall n \in \omega)(\exists s, t \in [\omega]^n)(s \times t \subset A)$.

Examples

Ideals $\mathcal{ED}_{\mathrm{fin}}$, $\mathrm{Fin} \times \mathrm{Fin}$ and $\mathcal{I}_{\mathcal{H}^2}$ are TRS.

For every ideal $\mathcal{I}$ on ω. Let $r_{\mathcal{I}}$ be the least natural number k such that $(\forall m)(\forall c : [\omega]^2 \to m)(\exists X \in \mathcal{I}^+)(|c''[X]^2| = k)$.

Proposition

If $\mathcal{I}, \mathcal{J}$ are ideals on ω such that $\mathcal{I} \leq_K \mathcal{J}$, then $r_{\mathcal{I}} \leq r_{\mathcal{J}}$.

So if $\mathcal{I}$ and $\mathcal{J}$ are Katětov equivalent, $r_{\mathcal{I}} = r_{\mathcal{J}}$.

i Let m be a natural number and $c : [\omega]^2 \to m$ be a coloring.

ii $(\exists f : \omega \to \omega)(\forall I \in \mathcal{I})(f^{-1}[I] \in \mathcal{J})$.

iii Let $\varphi : [\omega]^2 \to m$ be such that $\varphi(\{m, n\}) = c(\{f(m), f(n)\})$.

iv $(\exists Y \in \mathcal{J}^+)(|\varphi''[Y]^2| \leq r_{\mathcal{J}})$.

v $X = f[Y] \in \mathcal{I}^+$ and $|c''[X]^2| = |\varphi''[Y]^2| \leq r_{\mathcal{J}}$.

Therefore $r_{\mathcal{I}} \leq r_{\mathcal{J}}$.

Proposition(Laflamme)

If $\mathcal{I} = \mathcal{ED}_{\mathrm{fin}}$, $r_{\mathcal{I}} = 3$.

Proposition(Dobrinen,N)

If $\mathcal{I} = \mathcal{I}_{\mathcal{H}^2}$, $r_{\mathcal{I}} = 5$.

It is a consequence of last Proposition that ideals $\mathcal{ED}_{\mathrm{fin}}$ and $\mathcal{I}_{\mathcal{H}^2}$ are not Katetov equivalent.

Proposition(N)

If $\mathcal{I}$ is an ideal TRS and $\mathcal{G}$ is a $(\mathcal{P}(\omega)/\mathcal{I}, \subset)-$generic filter, then $\mathcal{P}(\omega)$ $\mathcal{I}$ forces a Ramsey ultrafilter $\mathcal{U}$. ($\mathcal{U} \leq_{\mathrm{RK}} \mathcal{G}$).

Proof(for $\mathcal{ED}_{\mathrm{fin}}$).

- For every $X \in \mathcal{G}$, let $D(X) = \{m \in \omega : (\exists n \in \omega)((m, n) \in X)\}$.
- Let $\mathcal{U}$ be the ultrafilter generated by the collection of $D(X)$ with $X \in \mathcal{G}$.
- Let $c : [\omega]^2 \to m$ be a coloring.
- Define $\mathcal{D} = \{X \in \mathcal{ED}^+_{\mathrm{fin}} : D(X)$ is c-homogeneous$\}$.

- Take $X \in [T]$.
- For every $i \in m$, let $\mathcal{F}_i = \{t \in T_2 : c(\pi(t)) = i\}$.
- By N-W Theorem there exists $i_0 \in m$ and $Y \in [T] \cap [X]^\omega$ such that $T_2 \cap [Y]^{<\omega} \subseteq \mathcal{F}_{i_0}$.
- Hence $\mathcal{D}$ is dense on $\mathcal{P}(\omega)/\mathcal{I}$.
- $Z \in \mathcal{G} \cap \mathcal{D}$ implies that $D(Z) \in \mathcal{U}$ is c-homogeneous.

Proposition

A consequence of the last Proposition is that the summable ideal is not a TRS.

(There are no rapid filters RK below the filter $(\mathcal{P}(\omega)/\mathcal{I}_{\frac{1}{n}}, \subseteq)$)-generic.

References

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