On n-fold sum of a non-flat continuum

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A function $f : \mathbb{R}^N \to \mathbb{R}$ is called an *additive functional* if

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}^N.$$

Fact (folklore)

An additive functional $f : \mathbb{R}^N \to \mathbb{R}$ upper bounded on a non-empty open set $T \subset \mathbb{R}^N$ is continuous.

For $n \in \mathbb{N}$ and $T \subset X$ denote by $T^{+n} := \underbrace{T + \cdots + T}_{n}$ n-fold sum of T.

Theorem (folklore)

An additive functional $f : \mathbb{R}^N \to \mathbb{R}$ upper bounded on a set $T \subset \mathbb{R}^N$ with

$$(S_n) \qquad \operatorname{int} T^{+n} \neq \emptyset$$

for some $n \in \mathbb{N}$ is continuous.

Steinhaus Theorem (1920)

For every sets $A, B \subset \mathbb{R}^N$ of positive Lebesgue measure $\operatorname{int}(A+B) \neq \emptyset$.

Pettis–Piccard Theorem (1939, 1951)

Let X be a topological group. For every non-meager sets $A, B \subset X$ with the Baire property $\operatorname{int}(A+B) \neq \emptyset$.

- Sets of positive Lebesgue measure in $\mathbb{R}^N$;
- non-meager sets with the Baire property in a topological group;
- the Cantor ternary set C in $\mathbb{R}$

have the property (S_2).

- H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fund. Math. 1 (1920), 99-104.
- S. Piccard, *Sur les ensembles de distances des ensembles de points d'un espace Euclidien*, Mem. Univ. Neuchâtel, vol. 13, Secrétariat Univ., Neuchâtel, 1939.
- B.J. Pettis, *Remarks on a theorem of E. J. McShane*, Proc. Amer. Math. Soc. 2 (1951), 166-171.

Theorem (Jabłoński, 1999)

If $U \subset \mathbb{R}^N$ is a non-empty open set and $f : U \to \mathbb{R}$ is a continuous non-affine function, then

$$\operatorname{int}(\mathrm{Gr}\, f + \mathrm{Gr}\, f) \neq \emptyset;$$

i.e. $\mathrm{Gr}\, f \subset \mathbb{R}^{N+1}$ has the property (S_2).

There are also sets satisfying (S_{n+1}) but not (S_n); e.g.
Let $T \subset \mathbb{R}^3$ be given by $T := T_x \cup T_y \cup T_z$, where

$$\begin{aligned} T_x &:= [0,1] \times \{0\} \times \{0\}, \\ T_y &:= \{0\} \times [0,1] \times \{0\}, \\ T_z &:= \{0\} \times \{0\} \times [0,1]. \end{aligned}$$

Then $\lambda_3(T+T) = 0$ and $\operatorname{int}(T+T+T) \neq \emptyset$.

- W. Jabłoński, *On a class of sets connected with a convex function*, Abh. Math. Semin. Univ. Hamburg 69 (1999), 205–210.

Theorem (Banakh–Bartoszewicz–Filipczak–Szymonik, 2015)

Let $k \geqslant 2$ and $\alpha \in \left[\frac{1}{k+1}, \frac{1}{k}\right)$. For the Cantor-type set $C_\alpha \subset \mathbb{R}$ with self–similarity α

$$\lambda_1(C_\alpha^{+(k-1)}) = 0 \ \text{ and } \ C_\alpha^{+k} = [0, k];$$

i.e. C_α satisfies (S_k) but not (S_{k-1}).

- T. Banakh, A. Bartoszewicz, M. Filipczak, E. Szymonik, *Topological and measure properties of some self-similar sets*, Topol. Methods Nonlinear Anal. 46 (2015), 1013–1028.

Theorem (Ger, 1973)

Let $I \subset \mathbb{R}$ be an interval, $n \geqslant 2$ and $\varphi : I \to \mathbb{R}^n$ be a C^1-function such that $\frac{\partial(\varphi_1,\ldots,\varphi_n)}{\partial(x_1,\ldots,x_n)} := \begin{vmatrix} \varphi_1'(x_1) & \ldots & \varphi_1'(x_n) \\ \vdots & \ddots & \vdots \\ \varphi_n'(x_1) & \ldots & \varphi_n'(x_n) \end{vmatrix} \neq 0$ for almost all $(x_1, \ldots, x_n)$ in I^n.

If $Z \subset I$ is a set of positive Lebesgue measure, then

$$\lambda_n(\varphi(Z)^{+n}) > 0 \ \text{ and } \ \operatorname{int} \varphi(Z)^{+2n} \neq \emptyset$$

(i.e. $\varphi(Z) \subset \mathbb{R}^n$ has (S_{2n}) property).

What about a topological counterpart of Ger's result?

- R. Ger, *Thin sets and convex functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 413–416.

Algebraic sum of n continua in $\mathbb{R}^n$

Definition 1

A subset A of a real topological vector space X is called:

- *flat* if the affine hull of A is nowhere dense in X;
- *nowhere flat* if each non-empty relatively open subset $U \subset A$ is not flat in X.

If $\dim X = n$, then a set $A \subset X$ is flat if and only if A is contained in a hyperplane.

$$\text{nowhere flatness} \Longrightarrow \text{non-flatness}$$
$$\text{non-flatness} \not\Longrightarrow \text{nowhere flatness}$$

e.g. $T \subset \mathbb{R}^3$ given by $T := T_x \cup T_y \cup T_z$, where

$$T_x := [0,1] \times \{0\} \times \{0\},$$
$$T_y := \{0\} \times [0,1] \times \{0\},$$
$$T_z := \{0\} \times \{0\} \times [0,1].$$

Example 1

Let $\varphi : \mathbb{R} \to \mathbb{R}^3$ be given by $\varphi(t) := (\cos t, \sin t, t)$ for $t \in \mathbb{R}$. Then $\varphi(\mathbb{R}) \subset \mathbb{R}^3$ is nowhere flat.

/By a *continuum* we understand a connected compact metrizable space./

Theorem (Banakh–J.–Jabłoński, 2018)

Let $K_1, \ldots, K_n$ be continua in $\mathbb{R}^n$ containing the origin of $\mathbb{R}^n$. Assume that each continuum K_i contains a point e_i such that the vectors $e_1, \ldots, e_n$ are linearly independent. Then the algebraic sum $K := K_1 + \cdots + K_n$ has non-empty interior in $\mathbb{R}^n$ and $\lambda_n(K)$ is not smaller than the volume of the parallelotope $P := [0,1] \cdot e_1 + \cdots + [0,1] \cdot e_n$.

By Theorem (with $n = 2$) we obtain

Proposition (Kallman–Simmons, 1985)

If K is a continuum in the plane which does not lie on a line, then $\operatorname{int}(K - K) \neq \emptyset$.

- T. Banakh, E. Jabłońska, W. Jabłoński, *The continuity of additive and convex functions which are upper bounded on non-flat continua in* $\mathbb{R}^n$, Topol. Methods Nonlinear Anal. (accepted)
- R.R. Kallman, F.W. Simmons, *A theorem on planar continua and an application to authomorphisms of the field of complex numbers*, Topology Appl. 20 (1985), 251–255.

Corollary (Banakh–J.–Jabłoński, 2018)

For a continuum $K \subset \mathbb{R}^n$ the following conditions are equivalent:

(1) K^{+n} has non-empty interior in $\mathbb{R}^n$;

(2) K^{+n} has positive Lebesgue measure in $\mathbb{R}^n$;

(3) K is not flat in $\mathbb{R}^n$;

(4) $\operatorname{int} f(K) \neq \emptyset$ for any linear continuous functional $f : \mathbb{R}^n \to \mathbb{R}$, $f \neq 0$;

(5) each additive functional $f : \mathbb{R}^n \to \mathbb{R}$ upper bounded on K is continuous;

(6) each additive functional $f : \mathbb{R}^n \to \mathbb{R}$ bounded on K is continuous.

Problem (Banakh–J.–Jabłoński, 2018)

Is there a compact subset $K \subset \mathbb{R}^2$ such that $K + K$ has empty interior in $\mathbb{R}^2$ but for any non-zero linear continuous functional $f : \mathbb{R}^2 \to \mathbb{R}$ the image $f(K)$ has non-empty interior in $\mathbb{R}$?

Collectively nowhere flat subsets in $\mathbb{R}^n$

Definition 2

Subsets $A_1, \ldots, A_n$ of $\mathbb{R}^n$ are called *collectively nowhere flat in* $\mathbb{R}^n$ if any non-empty relatively open subsets $U_1 \subset A_1, \ldots, U_n \subset A_n$ contain points $a_1, b_1 \in U_1, \ldots, a_n, b_n \in U_n$ such that the vectors $b_1 - a_1, \ldots, b_n - a_n$ form a basis of the linear space $\mathbb{R}^n$.

For example, for a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ with $n \geqslant 2$:

- the closed intervals $[0,1] \cdot e_1, \ldots, [0,1] \cdot e_n$ are collectively nowhere flat;
- for each $i \in \{1, \ldots, n\}$ the set $[0,1] \cdot e_i$ is flat.

Proposition (Banakh–J.–Jabłoński, 2018)

A subspace $A \subset \mathbb{R}^n$ is nowhere flat in $\mathbb{R}^n$ if and only if the sequence of n its copies $A_1 = A, \ldots, A_n = A$ is collectively nowhere flat in $\mathbb{R}^n$.

Theorem (Banakh–J.–Jabłoński, 2018)

Let $K_1, \ldots, K_n$ be collectively nowhere flat locally connected subspaces of $\mathbb{R}^n$. For every non-meager subsets $B_1, \ldots, B_n$ in $K_1, \ldots, K_n$ the algebraic sum $B_1 + \cdots + B_n$ is non-meager in $\mathbb{R}^n$.

Corollary (Banakh–J.–Jabłoński, 2018)

Let K be a nowhere flat locally connected subset of $\mathbb{R}^n$ and A be a non-meager analytic subspace of K. Then A^{+n} is a non-meager analytic subset of $\mathbb{R}^n$ and $\operatorname{int} A^{+2n} \neq \emptyset$. Moreover, $0 \in \operatorname{int}(A - A)^{+n}$.

The condition of (collective) nowhere flatness in Theorem and Corollary is essential**!**

Example 2

- Let C be the Cantor ternary set in $[0,1]$.
- Let $f : C \to [0,1]$ be a continuous function given by

$$f\left(\sum_{n=1}^{\infty} \frac{2x_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

- Let $g : [0,1] \to [0,1]$ be the Cantor function, i.e. the unique monotone function extending f.
- Let Γ_f, Γ_g be the graphs of f and g, respectively.
- The set $K := \Gamma_g$ is connected but not nowhere flat in the plane $\mathbb{R}^2$.
- The sets $K_1 := K$, $K_2 := K$ are not collectively nowhere flat in $\mathbb{R}^2$.
- The set $A := \Gamma_g \setminus \Gamma_f$ is open (consequently non-meager and analytic) in $K = \Gamma_g$.
- $A + A$ is meager in $\mathbb{R}^2$.

A topological counterpart of Ger's Theorem:

Corollary (Banakh–J.–Jabłoński, 2018)

Let $n \in \mathbb{N}$. For any non-flat continuum $K \subset \mathbb{R}^n$ $\operatorname{int} K^{+n} \neq \emptyset$. Moreover, if K is locally connected and nowhere flat in $\mathbb{R}^n$, then for any non-meager analytic subspace A of K A^{+n} is a non-meager analytic subset of $\mathbb{R}^n$ and $\operatorname{int} A^{+2n} \neq \emptyset$.

Corollary (Banakh–J.–Jabłoński, 2018)

Let $I \subset \mathbb{R}$ be an interval, $n \geqslant 2$ and $\varphi : I \to \mathbb{R}^n$ be a C^1-function such that $\frac{\partial(\varphi_1,\ldots,\varphi_n)}{\partial(x_1,\ldots,x_n)} \neq 0$ for almost all $(x_1, \ldots, x_n)$ in I^n (then $K := \varphi(I)$ is locally connected and nowhere flat in $\mathbb{R}^n$).
If $Z \subset I$ is a non-meager set with the Baire property, then $\varphi(Z)^{+n}$ is a non-meager analytic set in $\mathbb{R}^n$ and

$$\operatorname{int} \varphi(Z)^{+2n} \neq \emptyset.$$

- T. Banakh, E. Jabłońska, W. Jabłoński, *The continuity of additive and convex functions which are upper bounded on non-flat continua in* $\mathbb{R}^n$, Topol. Methods Nonlinear Anal. (accepted), arXiv:1805.01997v2 [math.GN]