Generic structures

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Part 1

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- a partial associative composition operation $\circ$ defined on arrows, where $f\circ g$ is defined $\iff$ the domain of g coincides with the domain of f.

Furthermore, for each $A\in\mathrm{Obj}(\mathfrak{K})$ there is an *identity* $\mathrm{id}_A\in\mathfrak{K}(A,A)$ satisfying $\mathrm{id}_A\circ g=g$ and $f\circ\mathrm{id}_A=f$ for $f\in\mathfrak{K}(A,X)$, $g\in\mathfrak{K}(Y,A)$, $X,Y\in\mathrm{Obj}(\mathfrak{K})$.

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Let $\vec{x}$ be a sequence in $\mathfrak{K}$. The colimit of $\vec{x}$ is a pair $\langle X, \{x_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $x_n^\infty \colon X_n \to X$ satisfying:

1. $x_n^\infty = x_m^\infty \circ x_n^m$ for every $n < m$.
2. If $\langle Y, \{y_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $y_n^\infty \colon X_n \to Y$ satisfies $y_n^\infty = y_m^\infty \circ y_n^m$ for every $n < m$ then there is a unique arrow $f \colon X \to Y$ satisfying $f \circ x_n^\infty = y_n^\infty$ for every $n \in \mathbb{N}$.

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More generally, after Odd's move finishing with an object A_{2k-1}, Eve chooses $A_{2k} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k-1}^{2k} \colon A_{2k-1} \to A_{2k}$.

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Next, Odd chooses $A_{2k+1} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k}^{2k+1} \colon A_{2k} \to A_{2k+1}$. And so on...

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The result of a play is a sequence $\vec{a}$:

$$A_0 \xrightarrow{a_0^1} A_1 \longrightarrow \cdots \longrightarrow A_{2k-1} \xrightarrow{a_{2k-1}^{2k}} A_{2k} \longrightarrow \cdots$$

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We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-generic if Odd has a strategy in the Banach-Mazur game $\mathrm{BM}(\mathfrak{K})$ such that the colimit of the resulting sequence $\vec{a}$ is always isomorphic to U, no matter how Eve plays.

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Proof.

The rules for Eve and Odd are the same. □

Examples

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Example

Let $\mathfrak{K}$ be the category of all finite acyclic graphs with embeddings.
Then the countable everywhere infinitely branching tree is $\mathfrak{K}$-generic.

Theorem (Urysohn, 1927)

There exists a unique Polish metric space $\mathbb{U}$ with the following property:

(E) *For every finite metric spaces $A \subseteq B$, every isometric embedding $e \colon A \to \mathbb{U}$ can be extended to an isometric embedding $f \colon B \to \mathbb{U}$.*

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Theorem

Let $\mathfrak{M}_{\mathrm{fin}}$ be the category of finite metric spaces with isometric embeddings.
Then the Urysohn space $\mathbb{U}$ is $\mathfrak{M}_{\mathrm{fin}}$-generic.

The amalgamation property

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Definition

We say that $\mathfrak{K}$ has amalgamations at $Z \in \mathrm{Obj}(\mathfrak{K})$ if for every $\mathfrak{K}$-arrows $f\colon Z \to X$, $g\colon Z \to Y$ there exist $\mathfrak{K}$-arrows $f'\colon X \to W$, $g'\colon Y \to W$ such that $f' \circ f = g' \circ g$.

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We say that $\mathfrak{K}$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \mathrm{Obj}(\mathfrak{K})$.

Theorem (Universality)

Assume $\mathfrak{K}$ has the AP and U is $\mathfrak{K}$-generic.
Then for every $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{K}$, there exists an arrow

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Example

Let $\mathfrak{K}$ be the category of all finite linear graphs with embeddings. Then $\langle \mathbb{Z}, E \rangle$ is $\mathfrak{K}$-generic, where $xEy \iff |x - y| = 1$.
On the other hand, $\langle \mathbb{Z}, E \rangle \oplus \langle \mathbb{Z}, E \rangle \not\to \langle \mathbb{Z}, E \rangle$.

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2. For every $n \in \omega$, for every $\mathfrak{K}$-arrow $f\colon U_n \to Y$ there are $m > n$ and a $\mathfrak{K}$-arrow $g\colon Y \to U_m$ such that $g \circ f = u_n^m$.

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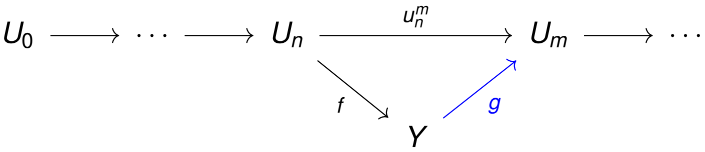

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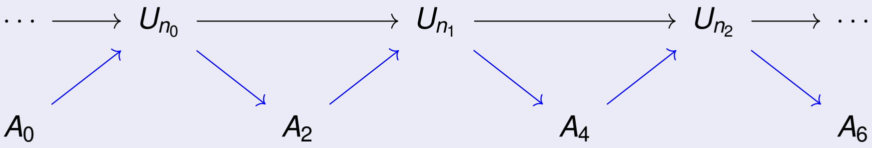

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Fraïssé categories

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A Fraïssé category is a countable category $\mathfrak{K}$ satisfying:

1. For every $X, Y \in \mathrm{Obj}(\mathfrak{K})$ there is $U \in \mathrm{Obj}(\mathfrak{K})$ such that

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2. $\mathfrak{K}$ has the amalgamation property.

Theorem 2

Assume $\mathfrak{K} \subseteq \mathfrak{L}$ is such that every sequence in $\mathfrak{K}$ converges in $\mathfrak{L}$ and $\mathfrak{K}$ is a Fraïssé category. Then there exists a $\mathfrak{K}$-generic object in $\mathfrak{L}$.

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Let

$$\mathscr{D} = \{D_{n,f} \colon n \in \omega,\ f \in \mathfrak{K}\} \cup \{E_{n,A} \colon n \in \omega,\ X \in \operatorname{Obj}(\mathfrak{K})\},$$

where

$$D_{n,f} = \{\vec{x} \in \mathbb{P} \colon X_n = \operatorname{dom}(f) \implies (\exists m > n)(\exists g)\ g \circ f = x_n^m\},$$

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Let $\vec{u}$ be the sequence coming from a $\mathscr{D}$-generic filter/ideal. Then $\vec{u}$ is Fraïssé, therefore $U = \lim \vec{u}$ is $\mathfrak{K}$-generic. □

Fraïssé theory

Definition

A Fraïssé class is a class of finite models of a fixed countable language satisfying:

(H) For every $A \in \mathscr{F}$, every model isomorphic to a submodel of A is in $\mathscr{F}$.

(JEP) Every two models from $\mathscr{F}$ embed into a single model from $\mathscr{F}$.

(AP) $\mathscr{F}$ has the amalgamation property for embeddings.

(CMT) $\mathscr{F}$ has countably many isomorphic types.

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Conversely, if U is a countable homogeneous model then the class of all models isomorphic to finite submodels of U is Fraïssé.

More examples

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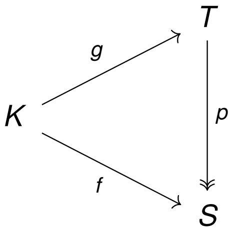

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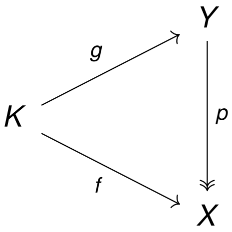

Theorem (Bielas, Walczyńska, K.)

Let 2^ω denote the Cantor set. A continuous mapping $\eta \colon K \to 2^\omega$ is $\mathfrak{K}_K$-generic $\iff$ η is a topological embedding and $\eta[K]$ is nowhere dense in 2^ω.

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Corollary (Knaster & Reichbach 1953)

Let $h\colon A \to B$ be a homeomorphism between closed nowhere dense subsets of 2^ω. Then there exists a homeomorphism $H\colon 2^\omega \to 2^\omega$ such that

$$H \restriction A = h.$$

The Gurarii space

Theorem (Gurarii 1966)

There exists a separable Banach space $\mathbb{G}$ with the following property.

(G) *For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every linear isometric embedding $e \colon E \to \mathbb{G}$ there exists a linear ε-isometric embedding $f \colon F \to \mathbb{G}$ such that $f \restriction E = e$.*

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Elementary proof: Solecki & K. 2013.

Theorem

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Key Lemma (Solecki & K.)

Let X, Y be finite-dimensional normed spaces, let $f\colon X \to Y$ be an ε-isometry with $0 < \varepsilon < 1$. Then there exist a finite-dimensional normed space Z and isometric embeddings $i\colon X \to Z$, $j\colon Y \to Z$ such that

$$\|i - j \circ f\| \leqslant \varepsilon.$$

The pseudo-arc

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The pseudo-arc is $\mathfrak{I}$-generic.

Part 2

Weak Fraïssé sequences

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Definition

A sequence $\vec{u}\colon \omega \to \mathfrak{K}$ is a weak Fraïssé sequence if it satisfies the following conditions:

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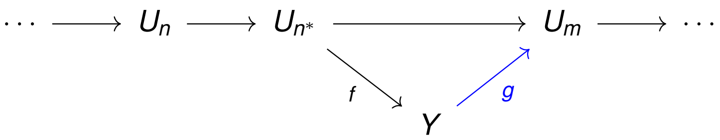
1. For every $A \in \mathrm{Obj}(\mathfrak{K})$ there is n such that $\mathfrak{K}(A, U_n) \neq \emptyset$.
2. For every $n \in \omega$ there exists $n^* > n$ such that for every $\mathfrak{K}$-arrow $f\colon U_{n^*} \to Y$ there are $m > n^*$ and a $\mathfrak{K}$-arrow $g\colon Y \to U_m$ with $g \circ f \circ u_n^{n^*} = u_n^m$.



Theorem

Assume $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$ and $U = \lim \vec{u}$. Then U is $\mathfrak{K}$-generic.

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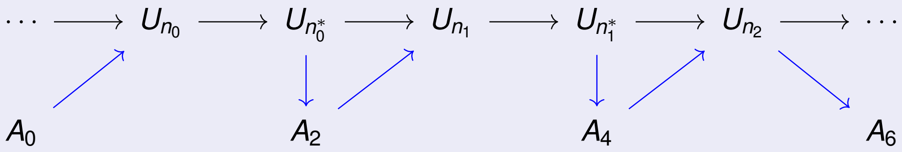
Proof.



□

Weakenings of amalgamation

Definition

We say that $\mathfrak{K}$ has the cofinal amalgamation property (CAP) if for every $Z \in \mathrm{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e\colon Z \to Z'$ such that $\mathfrak{K}$ has amalgamations at Z'.

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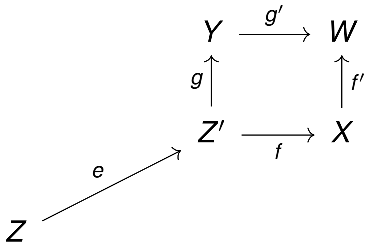
Definition (Ivanov 1999; Kechris & Rosendal 2007; Kruckman 2016)

We say that $\mathfrak{K}$ has the weak amalgamation property (WAP) if for every $Z \in \mathrm{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e \colon Z \to Z'$ such that for every $\mathfrak{K}$-arrows $f \colon Z' \to X$, $g \colon Z' \to Y$ there exist $\mathfrak{K}$-arrows $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f \circ e = g' \circ g \circ e$.

CAP and WAP



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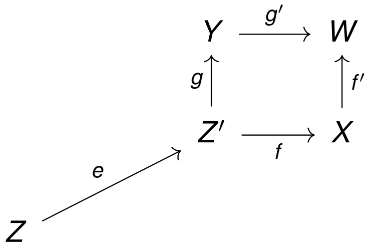

Proposition

Finite graphs of vertex degree $\leqslant 2$ have the CAP.

Weak injectivity

Definition

An object $V \in \mathrm{Obj}(\mathfrak{L})$ is weakly $\mathfrak{K}$-injective if

- every $\mathfrak{K}$-object has an $\mathfrak{L}$-arrow into V, and
- for every $\mathfrak{L}$-arrow $e \colon A \to V$ there exists a $\mathfrak{K}$-arrow $i \colon A \to B$ such that for every $\mathfrak{K}$-arrow $f \colon B \to Y$ there is an $\mathfrak{L}$-arrow $g \colon Y \to V$ satisfying $g \circ f \circ i = e$.

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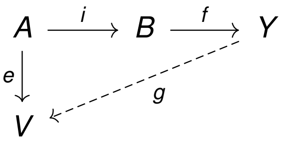

Theorem (Krawczyk & K. 2016)

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Theorem (Krawczyk & K. 2016)

Let $\mathfrak{K}$ be as above and let U be a countably generated model. The following properties are equivalent:

(a) *U is $\mathfrak{K}$-generic.*

(b) *Eve does not have a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$.*

(c) *U is weakly $\mathfrak{K}$-injective.*

The first example of a weak Fraïssé class with no CAP

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- J.-F. PABION, *Relations préhomogènes*, C. R. Acad. Sci. Paris Sér. A-B 274 (1972) A529–A531.

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A quote from Pabion's paper:

3° M. Pouzèt m'a communiqué l'exemple suivant de relation uniformément préhomogène et non pseudo-homogène. Sur Q, définir R (x, y, z) par $x < y$, $x < z$ et $y \neq z$.

(*) Séance du 7 février 1972.
(1) J. P. CALAIS, *Comptes rendus*, 265, série A, 1967, p. 2.
(2) R. FRAÏSSÉ, *Cours de Logiques mathématiques*, I, Gauthiers-Villars, Paris, 1967, deuxième édition 1971.
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Mathématiques,
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69-*Villeurbanne, Rhône.*

Weak Fraïssé theory

Definition

A weak Fraïssé class is a class $\mathscr{F}$ of finitely generated models of a fixed countable signature, closed under isomorphisms, having with many types, satisfying (JEP) and (WAP).

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Theorem

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Conversely, given a countable weakly homogeneous model M, its age

$$\mathscr{F} = \{A \colon A \text{ is finitely generated and embeddable into } M\}$$

is a weak Fraïssé class.

Definition

A structure M is weakly homogeneous if for every finitely generated substructure $A \subseteq M$ there is a bigger finitely generated substructure $B \subseteq M$ containing A such that every embedding $e\colon A \to M$ extendable to B extends to an automorphism of M.

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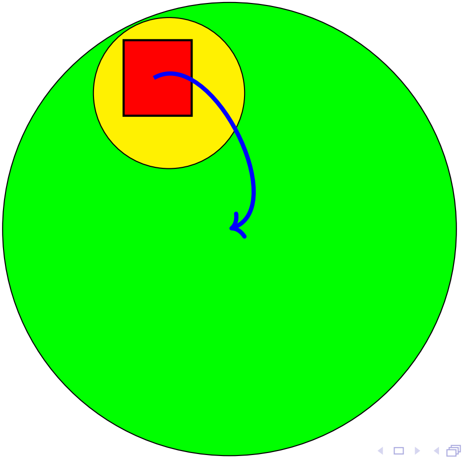

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2. A. KRAWCZYK, W. KUBIŚ, *Games on finitely generated structures*, preprint, arXiv:1701.05756

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Theorem (Krawczyk, Kruckman, Panagiotopoulos, K. 2018)

There exist continuum many hereditary weak Fraïssé classes of finite graphs without the cofinal AP.

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Example

Let $\mathscr{G}$ be the class of all finite acyclic graphs in which no two vertices of degree > 2 are adjacent.
Then $\mathscr{G}$ is a weak Fraïssé class failing the CAP.

Peano continua

Theorem (Bartoš, K. 2018)

Let $\mathfrak{K}$ be a class of non-degenerate Peano continua treated as a category with all continuous surjections. Then the pseudo-arc is $\mathfrak{K}^{\mathrm{op}}$-generic.

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Theorem (Kwiatkowska, K. 2017)

The Poulsen simplex is generic over the (opposite) category of finite-dimensional simplices with affine surjections.

Question (Eric Jaligot, 2007)

Let M be a countable homogeneous structure. Is it always true that the group $\mathrm{Aut}(M)$ contains isomorphic copies of all groups of the form $\mathrm{Aut}(X)$, where X is a substructure of M?

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Let M be a countable homogeneous structure. Is it always true that the group $\mathrm{Aut}(M)$ contains isomorphic copies of all groups of the form $\mathrm{Aut}(X)$, where X is a substructure of M?

If this is the case, we shall say that $\mathrm{Aut}(M)$ is universal.

Uniform homogeneity

Definition (Kuzeljević, K. 2018)

A structure M is uniformly homogeneous if

1. M is homogeneous and
2. for every finite substructure $A \subseteq M$ there exists an extension operator $e_A\colon \operatorname{Aut}(A) \to \operatorname{Aut}(M)$ such that

$$e_A(g \circ h) = e_A(g) \circ e_A(h)$$

for every $g, h \in \operatorname{Aut}(A)$.

A homogeneous digraph that is not uniformly homogeneous

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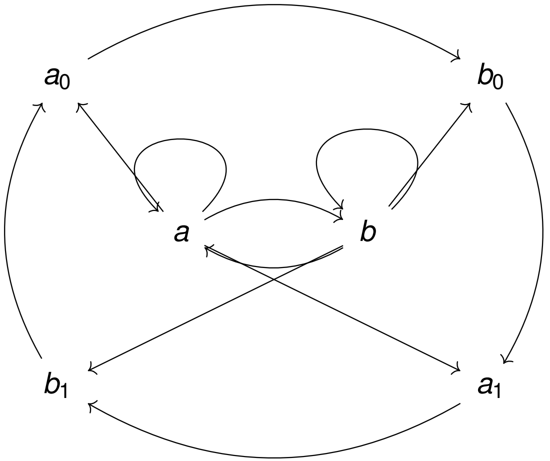

Katětov functors

Definition

Let $\mathscr{F}$ be a class of finite structures of the same type and let M be a countable homogeneous structure such that every $A \in \mathscr{F}$ embeds into M and every finite substructure of M is isomorphic to some $A \in \mathscr{F}$. A Katětov functor is a pair $\langle K, \eta \rangle$ such that K assigns to each embedding $e \colon A \to B$ with $A, B \in \mathscr{F}$ an embedding $K(e) \colon M \to M$, η assigns to each $A \in \mathscr{F}$ an embedding $\eta_A \colon A \to M$. Furthermore, K is a functor, i.e., $K(\mathrm{id}_A) = \mathrm{id}_M$, $K(e \circ f) = K(e) \circ K(f)$, and the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & M \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle K(e)} \\ B & \xrightarrow[\eta_B]{} & M \end{array}$$

for every embedding $e \colon A \to B$ with $A, B \in \mathscr{F}$.

Theorem (Mašulović & K.)

Assume $\langle \mathscr{F}, M \rangle$ admits a Katětov functor. Then for every substructure X of M there exists a topological group embedding

$$e_X\colon \operatorname{Aut}(X) \to \operatorname{Aut}(M).$$

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Claim (☹)

Most of the well known homogeneous relational structures admit a Katětov functor.

Non-universal automorphism groups

Theorem (Shelah & K. 2018)

There exists a countable homogeneous relational structure E such that:

- *every finite group embeds into* $\mathrm{Aut}(E)$,
- S_∞ *does not embed into* $\mathrm{Aut}(E)$,
- $S_\infty \approx \mathrm{Aut}(X)$ *for some* $X \subseteq E$.

Furthermore, E is not uniformly homogeneous.

Theorem (Shelah & K. 2018)

There exists a countable homogeneous relational structure M such that:

- $\mathrm{Aut}(M)$ *is torsion-free,*
- *for every $n \in \mathbb{N}$ there is a finite $A \subseteq M$ with $S_n \approx \mathrm{Aut}(A)$.*

Some more references

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