Generic structures

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Part 1

- Preliminaries
- Generic objects
- Fraïssé categories
- Fraïssé theory
- More examples

Part 2

- Weak Fraïssé theory
- Examples
- Automorphism groups
Part 1
Categories

A *category* $\mathcal{C}$ consists of

- a class of objects $\text{Obj}(\mathcal{C})$,
Categories

A *category* \( \mathcal{K} \) consists of

- a class of objects \( \text{Obj}(\mathcal{K}) \),
- a class of arrows \( \bigcup_{A,B \in \text{Obj}(\mathcal{K})} \mathcal{K}(A, B) \), where \( f \in \mathcal{K}(A, B) \) means \( A \) is the *domain* of \( f \) and \( B \) is the *codomain* of \( f \),

Furthermore, for each \( A \in \text{Obj}(\mathcal{K}) \) there is an *identity* \( \text{id}_A \in \mathcal{K}(A, A) \) satisfying \( \text{id}_A \circ g = g \) and \( f \circ \text{id}_A = f \) for \( f \in \mathcal{K}(A, X) \), \( g \in \mathcal{K}(Y, A) \), \( X, Y \in \text{Obj}(\mathcal{K}) \).
Categories

A *category* $\mathcal{K}$ consists of

- a class of objects $\text{Obj}(\mathcal{K})$,
- a class of arrows $\bigcup_{A,B \in \text{Obj}(\mathcal{K})} \mathcal{K}(A, B)$, where $f \in \mathcal{K}(A, B)$ means $A$ is the *domain* of $f$ and $B$ is the *codomain* of $f$,
- a partial associative composition operation $\circ$ defined on arrows, where $f \circ g$ is defined $\iff$ the domain of $g$ coincides with the domain of $f$. 

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- a class of objects $\text{Obj}(\mathcal{K})$,
- a class of arrows $\bigcup_{A,B \in \text{Obj}(\mathcal{K})} \mathcal{K}(A, B)$, where $f \in \mathcal{K}(A, B)$ means $A$ is the \textit{domain} of $f$ and $B$ is the \textit{codomain} of $f$,
- a partial associative composition operation $\circ$ defined on arrows, where $f \circ g$ is defined $\iff$ the domain of $g$ coincides with the domain of $f$.

Furthermore, for each $A \in \text{Obj}(\mathcal{K})$ there is an \textit{identity} $\text{id}_A \in \mathcal{K}(A, A)$ satisfying $\text{id}_A \circ g = g$ and $f \circ \text{id}_A = f$ for $f \in \mathcal{K}(A, X)$, $g \in \mathcal{K}(Y, A)$, $X, Y \in \text{Obj}(\mathcal{K})$. 
Definition

A **sequence** in $\mathcal{K}$ is a functor $\vec{x}$ from $\omega$ into $\mathcal{K}$. 

Let $\vec{x}$ be a sequence in $\mathcal{K}$. The colimit of $\vec{x}$ is a pair $\langle X, \{x_\infty^n\}_{n \in \mathbb{N}} \rangle$ with $x_\infty^n : X^n \to X$ satisfying:

1. $x_\infty^n = x_\infty^m \circ x^m_n$ for every $n < m$.

2. If $\langle Y, \{y_\infty^n\}_{n \in \mathbb{N}} \rangle$ with $y_\infty^n : X^n \to Y$ satisfies $y_\infty^n = y_\infty^m \circ y^m_n$ for every $n < m$ then there is a unique arrow $f : X \to Y$ satisfying $f \circ x_\infty^n = y_\infty^n$ for every $n \in \mathbb{N}$. 

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Definition

A sequence in $\mathcal{K}$ is a functor $\vec{x}$ from $\omega$ into $\mathcal{K}$.

\[ X_0 \xrightarrow{x_0^1} X_1 \xrightarrow{x_1^2} X_2 \xrightarrow{x_2^3} \ldots \]
Definition

A sequence in \( K \) is a functor \( \vec{x} \) from \( \omega \) into \( K \).

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X_0 \xrightarrow{x_0^1} X_1 \xrightarrow{x_1^2} X_2 \xrightarrow{x_2^3} \ldots
\]

Definition

Let \( \vec{x} \) be a sequence in \( K \). The colimit of \( \vec{x} \) is a pair \( \langle X, \{ x_n^\infty \}_{n \in \mathbb{N}} \rangle \) with \( x_n^\infty : X_n \to X \) satisfying:

1. \( x_n^\infty = x_m^\infty \circ x_n^m \) for every \( n < m \).
2. If \( \langle Y, \{ y_n^\infty \}_{n \in \mathbb{N}} \rangle \) with \( y_n^\infty : X_n \to Y \) satisfies \( y_n^\infty = y_m^\infty \circ y_n^m \) for every \( n < m \) then there is a unique arrow \( f : X \to Y \) satisfying \( f \circ x_n^\infty = y_n^\infty \) for every \( n \in \mathbb{N} \).
The Banach-Mazur game

**Definition**

The *Banach-Mazur game* $BM(K)$ played on $K$ is described as follows.

There are two players: Eve and Odd. Eve starts by choosing $A_0 \in \text{Obj}(K)$. Then Odd chooses $A_1 \in \text{Obj}(K)$ together with a $K$-arrow $a_{01} : A_0 \to A_1$. More generally, after Odd's move finishing with an object $A_{2^k-1}$, Eve chooses $A_{2^k} \in \text{Obj}(K)$ together with a $K$-arrow $a_{2^k-1,2^k} : A_{2^k-1} \to A_{2^k}$. Next, Odd chooses $A_{2^k+1} \in \text{Obj}(K)$ together with a $K$-arrow $a_{2^k+1,2^k+2} : A_{2^k+1} \to A_{2^k+2}$. And so on...

The result of a play is a sequence $\vec{a} : A_0 \to A_1 \to \cdots \to A_{2^k-1} \to A_{2^k} \to \cdots$.
The Banach-Mazur game

**Definition**

The **Banach-Mazur game** BM(\(K\)) played on \(K\) is described as follows. There are two players: *Eve* and *Odd*. Eve starts by choosing \(A_0 \in \text{Obj}(K)\).
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The Banach-Mazur game $BM(\mathcal{K})$ played on $\mathcal{K}$ is described as follows. There are two players: Eve and Odd. Eve starts by choosing $A_0 \in \text{Obj}(\mathcal{K})$. Then Odd chooses $A_1 \in \text{Obj}(\mathcal{K})$ together with a $\mathcal{K}$-arrow $a^1_0 : A_0 \to A_1$. And so on...
The Banach-Mazur game

**Definition**

The Banach-Mazur game BM (K) played on K is described as follows. There are two players: Eve and Odd. Eve starts by choosing \( A_0 \in \text{Obj}(K) \). Then Odd chooses \( A_1 \in \text{Obj}(K) \) together with a K-arrow \( a_0^1 : A_0 \to A_1 \). More generally, after Odd’s move finishing with an object \( A_{2k-1} \), Eve chooses \( A_{2k} \in \text{Obj}(K) \) together with a K-arrow \( a_{2k-1}^{2k} : A_{2k-1} \to A_{2k} \).
The Banach-Mazur game

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The Banach-Mazur game $\text{BM}(\mathcal{K})$ played on $\mathcal{K}$ is described as follows. There are two players: Eve and Odd. Eve starts by choosing $A_0 \in \text{Obj}(\mathcal{K})$. Then Odd chooses $A_1 \in \text{Obj}(\mathcal{K})$ together with a $\mathcal{K}$-arrow $a_0^1 : A_0 \to A_1$. More generally, after Odd’s move finishing with an object $A_{2k-1}$, Eve chooses $A_{2k} \in \text{Obj}(\mathcal{K})$ together with a $\mathcal{K}$-arrow $a_{2k}^{2k-1} : A_{2k-1} \to A_{2k}$. Next, Odd chooses $A_{2k+1} \in \text{Obj}(\mathcal{K})$ together with a $\mathcal{K}$-arrow $a_{2k}^{2k+1} : A_{2k} \to A_{2k+1}$. And so on...
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The **Banach-Mazur game** BM($\mathcal{K}$) played on $\mathcal{K}$ is described as follows. There are two players: *Eve* and *Odd*. Eve starts by choosing $A_0 \in \text{Obj}(\mathcal{K})$. Then Odd chooses $A_1 \in \text{Obj}(\mathcal{K})$ together with a $\mathcal{K}$-arrow $a_0^1 : A_0 \to A_1$. More generally, after Odd’s move finishing with an object $A_{2k-1}$, Eve chooses $A_{2k} \in \text{Obj}(\mathcal{K})$ together with a $\mathcal{K}$-arrow $a_{2k-1}^{2k} : A_{2k-1} \to A_{2k}$. Next, Odd chooses $A_{2k+1} \in \text{Obj}(\mathcal{K})$ together with a $\mathcal{K}$-arrow $a_{2k}^{2k+1} : A_{2k} \to A_{2k+1}$. And so on...

The result of a play is a sequence $\vec{a}$:

\[
A_0 \xrightarrow{a_0^1} A_1 \xrightarrow{\cdot} A_{2k-1} \xrightarrow{a_{2k-1}^{2k}} A_{2k} \xrightarrow{\cdot} \quad \text{etc.}
\]
Generic objects

General assumption: $\mathcal{K} \subseteq \mathcal{L}$.

Definition: We say that $U \in \text{Obj}(\mathcal{L})$ is $\mathcal{K}$-generic if Odd has a strategy in the Banach-Mazur game $\text{BM}(\mathcal{K})$ such that the colimit of the resulting sequence $\vec{a}$ is always isomorphic to $U$, no matter how Eve plays.

Proposition: A $\mathcal{K}$-generic object, if exists, is unique up to isomorphism.

Proof. The rules for Eve and Odd are the same.
General assumption: $\mathcal{K} \subseteq \mathcal{L}$.

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We say that $U \in \text{Obj}(\mathcal{L})$ is $\mathcal{K}$-generic if Odd has a strategy in the Banach-Mazur game $BM(\mathcal{K})$ such that the colimit of the resulting sequence $\tilde{a}$ is always isomorphic to $U$, no matter how Eve plays.
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The rules for Eve and Odd are the same.
Examples
Example
Let $\mathcal{K}$ be the category of all finite linearly ordered sets with embeddings. Then $\langle \mathbb{Q}, < \rangle$ is $\mathcal{K}$-generic.

Example
Let $\mathcal{K}$ be the category of all finite graphs with embeddings. Then the Rado graph $\mathcal{R} = \langle \mathbb{N}, E_\mathcal{R} \rangle$ is $\mathcal{K}$-generic, where $k < n$ are adjacent if and only if the $k$th digit in the binary expansion of $n$ is one.

Example
Let $\mathcal{K}$ be the category of all finite acyclic graphs with embeddings. Then the countable everywhere infinitely branching tree is $\mathcal{K}$-generic.
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Example

Let $\mathcal{K}$ be the category of all finite acyclic graphs with embeddings. Then the countable everywhere infinitely branching tree is $\mathcal{K}$-generic.
Theorem (Urysohn, 1927)

There exists a unique Polish metric space $\mathbb{U}$ with the following property:

(E) For every finite metric spaces $A \subseteq B$, every isometric embedding $e: A \to \mathbb{U}$ can be extended to an isometric embedding $f: B \to \mathbb{U}$. 

Furthermore:
- Every separable metric space embeds into $\mathbb{U}$.
- Every isometry between finite subsets of $\mathbb{U}$ extends to a bijective isometry of $\mathbb{U}$.

Theorem

Let $M_{\text{fin}}$ be the category of finite metric spaces with isometric embeddings.

Then the Urysohn space $\mathbb{U}$ is $M_{\text{fin}}$-generic.
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Theorem

Let $\mathcal{M}_{\text{fin}}$ be the category of finite metric spaces with isometric embeddings. Then the Urysohn space $\mathbb{U}$ is $\mathcal{M}_{\text{fin}}$-generic.
The amalgamation property

Definition
We say that $K$ has amalgamations at $Z \in \text{Obj}(K)$ if for every $K$-arrows $f : Z \to X$, $g : Z \to Y$ there exist $K$-arrows $f' : X \to W$, $g' : Y \to W$ such that $f' \circ f = g' \circ g$.

We say that $K$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \text{Obj}(K)$. 

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The amalgamation property

Definition

We say that \( \mathbb{K} \) has \textbf{amalgamations at} \( Z \in \text{Obj}(\mathbb{K}) \) if for every \( \mathbb{K} \)-arrows \( f: Z \to X, \ g: Z \to Y \) there exist \( \mathbb{K} \)-arrows \( f': X \to W, \ g': Y \to W \) such that \( f' \circ f = g' \circ g \).

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & W \\
\uparrow{g} & & \uparrow{f'} \\
Z & \xrightarrow{f} & X
\end{array}
\]
The amalgamation property

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We say that $\mathcal{K}$ has **amalgamations at** $Z \in \text{Obj}(\mathcal{K})$ if for every $\mathcal{K}$-arrows $f : Z \to X$, $g : Z \to Y$ there exist $\mathcal{K}$-arrows $f' : X \to W$, $g' : Y \to W$ such that $f' \circ f = g' \circ g$.

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Y & \xrightarrow{g} & W \\
\uparrow g & & \uparrow f' \\
Z & \xrightarrow{f} & X
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We say that $\mathcal{K}$ has the **amalgamation property (AP)** if it has amalgamations at every $Z \in \text{Obj}(\mathcal{K})$. 
Theorem (Universality)

Assume $\mathcal{K}$ has the AP and $U$ is $\mathcal{K}$-generic. Then for every $X = \lim \bar{x}$, where $\bar{x}$ is a sequence in $\mathcal{K}$, there exists an arrow

$$e: X \to U.$$
**Theorem (Universality)**

Assume $\mathcal{R}$ has the AP and $U$ is $\mathcal{R}$-generic. Then for every $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathcal{R}$, there exists an arrow $e: X \to U$.

**Example**

Let $\mathcal{R}$ be the category of all finite linear graphs with embeddings. Then $\langle \mathbb{Z}, E \rangle$ is $\mathcal{R}$-generic, where $xEy \iff |x - y| = 1$. On the other hand, $\langle \mathbb{Z}, E \rangle \oplus \langle \mathbb{Z}, E \rangle \not\to \langle \mathbb{Z}, E \rangle$. 
Fraïssé sequences

**Definition**

A Fraïssé sequence in $\mathcal{K}$ is a sequence $\vec{u} : \omega \to \mathcal{K}$ satisfying the following conditions:

1. For every $A \in \text{Obj}(\mathcal{K})$ there is $n$ such that $\mathcal{K}(A, U_n) \neq \emptyset$.
2. For every $n \in \omega$, for every $\mathcal{K}$-arrow $f : U_n \to Y$ there are $m > n$ and a $\mathcal{K}$-arrow $g : Y \to U_m$ such that $g \circ f = u_{m,n}$.
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$$
\begin{array}{ccccccc}
U_0 & \longrightarrow & \cdots & \longrightarrow & U_n & \xrightarrow{u^m_n} & U_m & \longrightarrow & \cdots \\
& & & & f & \downarrow & & g & \\
& & & & & & Y & & \\
\end{array}
$$
Theorem 1

Let $\bar{u}$ be a Fraïssé sequence in $\mathcal{K}$ and let $U = \lim \bar{u}$. Then $U$ is $\mathcal{K}$-generic.
Theorem 1

Let \( \bar{u} \) be a Fraïssé sequence in \( \mathcal{K} \) and let \( U = \lim \bar{u} \). Then \( U \) is \( \mathcal{K} \)-generic.

Proof.

\[ \cdots \rightarrow U_{n_0} \rightarrow U_{n_1} \rightarrow U_{n_2} \rightarrow \cdots \]

\[ A_0 \rightarrow A_2 \rightarrow A_4 \rightarrow A_6 \]
A Fraïssé category is a countable category $\mathcal{K}$ satisfying:

1. For every $X, Y \in \text{Obj}(\mathcal{K})$ there is $U \in \text{Obj}(\mathcal{K})$ such that

\[ \mathcal{K}(X, U) \neq \emptyset \neq \mathcal{K}(Y, U). \]
Fraïssé categories

Definition

A Fraïssé category is a countable category $\mathbb{K}$ satisfying:

1. For every $X, Y \in \text{Obj}(\mathbb{K})$ there is $U \in \text{Obj}(\mathbb{K})$ such that

$$\mathbb{K}(X, U) \neq \emptyset \neq \mathbb{K}(Y, U).$$

2. $\mathbb{K}$ has the amalgamation property.
Theorem 2

Assume $\mathcal{K} \subseteq \mathcal{L}$ is such that every sequence in $\mathcal{K}$ converges in $\mathcal{L}$ and $\mathcal{K}$ is a Fraïssé category. Then there exists a $\mathcal{K}$-generic object in $\mathcal{L}$.
Theorem 2
Assume $\mathcal{K} \subseteq \mathcal{L}$ is such that every sequence in $\mathcal{K}$ converges in $\mathcal{L}$ and $\mathcal{K}$ is a Fraïssé category. Then there exists a $\mathcal{K}$-generic object in $\mathcal{L}$.

Proof.
Let $\mathbb{P}$ be the poset of all finite sequences in $\mathcal{K}$, i.e., covariant functors from some $n \in \omega$ into $\mathcal{K}$. The ordering is end-extension.
Theorem 2
Assume $\mathcal{K} \subseteq \mathcal{L}$ is such that every sequence in $\mathcal{K}$ converges in $\mathcal{L}$ and $\mathcal{K}$ is a Fraïssé category. Then there exists a $\mathcal{K}$-generic object in $\mathcal{L}$.

Proof.
Let $\mathbb{P}$ be the poset of all finite sequences in $\mathcal{K}$, i.e., covariant functors from some $n \in \omega$ into $\mathcal{K}$. The ordering is end-extension.
Let
$$\mathcal{D} = \{D_{n,f} : n \in \omega, f \in \mathcal{K}\} \cup \{E_{n,A} : n \in \omega, X \in \text{Obj}(\mathcal{K})\},$$
where
$$D_{n,f} = \{\vec{x} \in \mathbb{P} : X_n = \text{dom}(f) \implies (\exists m > n)(\exists g) g \circ f = x_n^m\},$$
$$E_{n,A} = \{\vec{x} \in \mathbb{P} : (\exists m \geq n) \mathcal{K}(A, X_m) \neq \emptyset\}.$$
Theorem 2

Assume $\mathcal{K} \subseteq \mathcal{L}$ is such that every sequence in $\mathcal{K}$ converges in $\mathcal{L}$ and $\mathcal{K}$ is a Fraïssé category. Then there exists a $\mathcal{K}$-generic object in $\mathcal{L}$.

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Let $\mathbb{P}$ be the poset of all finite sequences in $\mathcal{K}$, i.e., covariant functors from some $n \in \omega$ into $\mathcal{K}$. The ordering is end-extension.

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$$\mathcal{D} = \{ D_{n,f} : n \in \omega, f \in \mathcal{K} \} \cup \{ E_{n,A} : n \in \omega, X \in \text{Obj}(\mathcal{K}) \},$$

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$$D_{n,f} = \{ \vec{x} \in \mathbb{P} : X_n = \text{dom}(f) \implies (\exists m > n)(\exists g) g \circ f = x^m_n \} ,$$

$$E_{n,A} = \{ \vec{x} \in \mathbb{P} : (\exists m \geq n) \mathcal{K}(A, X_m) \neq \emptyset \} .$$

Let $\vec{u}$ be the sequence coming from a $\mathcal{D}$-generic filter/ideal. Then $\vec{u}$ is Fraïssé, therefore $U = \lim \vec{u}$ is $\mathcal{K}$-generic.
A Fraïssé class is a class of finite models of a fixed countable language satisfying:

(H) For every $A \in \mathcal{F}$, every model isomorphic to a submodel of $A$ is in $\mathcal{F}$.

(JEP) Every two models from $\mathcal{F}$ embed into a single model from $\mathcal{F}$.

(AP) $\mathcal{F}$ has the amalgamation property for embeddings.

(CMT) $\mathcal{F}$ has countably many isomorphic types.
Theorem (Fraïssé, 1954)

Let $\mathcal{F}$ be a Fraïssé class. Then there exists a unique, up to isomorphism, countable model $U$ such that

1. $\mathcal{F}$ consists of all isomorphic types of finite submodels of $U$,
2. every isomorphism of finite submodels of $U$ extends to an automorphism of $U$ (in other words, $U$ is ultra-homogeneous).

Conversely, if $U$ is a countable homogeneous model then the class of all models isomorphic to finite submodels of $U$ is Fraïssé.
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Conversely, if $U$ is a countable homogeneous model then the class of all models isomorphic to finite submodels of $U$ is Fraïssé.
More examples
The Cantor set

Fix a compact 0-dimensional space $K$. Define the category $K$ as follows. The objects are continuous mappings $f: K \to S$ with $S$ finite. An arrow from $f: K \to S$ to $g: K \to T$ is a surjection $p: T \to S$ satisfying $p \circ g = f$.
The Cantor set

Fix a compact 0-dimensional space $K$. Define the category $\mathcal{K}_K$ as follows.
The Cantor set

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Fix a compact 0-dimensional space $K$. Define the category $\mathbb{K}_K$ as follows.
The objects are continuous mappings $f : K \to S$ with $S$ finite. An arrow from $f : K \to S$ to $g : K \to T$ is a surjection $p : T \to S$ satisfying $p \circ g = f$. 
Fix a compact 0-dimensional space $K$. Define the category $\mathcal{K}_K$ as follows.
The objects are continuous mappings $f : K \to S$ with $S$ finite. An arrow from $f : K \to S$ to $g : K \to T$ is a surjection $p : T \to S$ satisfying $p \circ g = f$. 

![Diagram](http://www.math.cas.cz/kubis/)
Let $\mathcal{C}_K$ be the category whose objects are continuous mappings $f: K \rightarrow X$ with $X$ metrizable compact 0-dimensional.
Let $\mathcal{L}_K$ be the category whose objects are continuous mappings $f: K \to X$ with $X$ metrizable compact 0-dimensional. An $\mathcal{L}_K$-arrow from $f: K \to X$ to $g: K \to Y$ is a \textbf{continuous surjection} $p: Y \to X$ satisfying $p \circ g = f$. 

W.Kubiš (http://www.math.cas.cz/kubis/)
Let $\mathcal{L}_K$ be the category whose objects are continuous mappings $f: K \to X$ with $X$ metrizable compact 0-dimensional. An $\mathcal{L}_K$-arrow from $f: K \to X$ to $g: K \to Y$ is a continuous surjection $p: Y \to X$ satisfying $p \circ g = f$. 

\[ \begin{array}{c}
\text{Y} \\
\searrow p \\
\downarrow \\
\text{X} \\
\nearrow f
\end{array} \] 

\[ \begin{array}{c}
\text{K} \\
\searrow g
\end{array} \]
Theorem (Bielas, Walczyńska, K.)

Let $2^\omega$ denote the Cantor set. A continuous mapping $\eta : K \to 2^\omega$ is $K$-generic $\iff$ $\eta$ is a topological embedding and $\eta[K]$ is nowhere dense in $2^\omega$. 

Corollary (Knaster & Reichbach 1953)

Let $h : A \to B$ be a homeomorphism between closed nowhere dense subsets of $2^\omega$. Then there exists a homeomorphism $H : 2^\omega \to 2^\omega$ such that $H|_A = h$. 
Theorem (Bielas, Walczyńska, K.)

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Let $h : A \rightarrow B$ be a homeomorphism between closed nowhere dense subsets of $2^\omega$. Then there exists a homeomorphism $H : 2^\omega \rightarrow 2^\omega$ such that

$$H \upharpoonright A = h.$$
The Gurarii space

Theorem (Gurarii 1966)

There exists a separable Banach space \( G \) with the following property.

\( (G) \) For every \( \varepsilon > 0 \), for every finite-dimensional normed spaces \( E \subseteq F \), for every linear isometric embedding \( e : E \to G \) there exists a linear \( \varepsilon \)-isometric embedding \( f : F \to G \) such that \( f \upharpoonright E = e \).
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**Theorem (Gurarii 1966)**

There exists a separable Banach space $\mathbb{G}$ with the following property.

(G) For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every linear isometric embedding $e : E \to \mathbb{G}$ there exists a linear $\varepsilon$-isometric embedding $f : F \to \mathbb{G}$ such that $f \upharpoonright E = e$.

**Theorem (Lusky 1976)**

Among separable spaces, property (G) determines the space $\mathbb{G}$ uniquely up to linear isometries.
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The Gurarii space $G$ is generic over the category $\mathcal{B}_{fd}$ of finite-dimensional normed spaces with linear isometric embeddings.
**Theorem**

The Gurarii space $\mathbb{G}$ is generic over the category $\mathcal{B}_{fd}$ of finite-dimensional normed spaces with linear isometric embeddings.

**Key Lemma (Solecki & K.)**

Let $X$, $Y$ be finite-dimensional normed spaces, let $f: X \to Y$ be an $\varepsilon$-isometry with $0 < \varepsilon < 1$. Then there exist a finite-dimensional normed space $Z$ and isometric embeddings $i: X \to Z$, $j: Y \to Z$ such that

$$\|i - j \circ f\| \leq \varepsilon.$$
The pseudo-arc

Let \( \mathcal{I} \) be the category of all continuous surjections from the unit interval \([0, 1]\) onto itself.
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Let \( \mathcal{C} \) be the category of all chainable continua.
The pseudo-arc

Let $\mathcal{I}$ be the category of all continuous surjections from the unit interval $[0, 1]$ onto itself.
Let $\mathcal{C}$ be the category of all chainable continua.

**Theorem**

*The pseudo-arc is $\mathcal{I}$-generic.*
Part 2
Weak Fraïssé sequences

Definition

A sequence \( \vec{u} : \omega \to K \) is a weak Fraïssé sequence if it satisfies the following conditions:

1. For every \( A \in \text{Obj}(K) \) there is \( n \) such that \( K(A, U_n) \neq \emptyset \).

2. For every \( n \in \omega \) there exists \( n^\ast > n \) such that for every \( K \)-arrow \( f : U_n \to Y \) there are \( m > n^\ast \) and a \( K \)-arrow \( g : Y \to U_m \) with \( g \circ f \circ u_n^\ast n = u_m \).

\[ U_n \quad U_n^\ast \quad U_m \]

\[ Y \quad f \quad g \]
Definition

A sequence $\vec{u} : \omega \to \mathcal{K}$ is a weak Fraïssé sequence if it satisfies the following conditions:

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\[
\begin{array}{c}
\mathcal{K} \ (A, \ U_n) \\
\mathcal{K} \ (U_n \rightarrow U_n) \\
\mathcal{K} \ (U_m ightarrow Y) \\
\mathcal{K} \ (g \circ f \circ u_n = u_m)
\end{array}
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Weak Fraïssé sequences

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1. For every $A \in \text{Obj}(\mathcal{K})$ there is $n$ such that $\mathcal{K}(A, U_n) \neq \emptyset$.
2. For every $n \in \omega$ there exists $n^* > n$ such that for every $\mathcal{K}$-arrow $f : U_{n^*} \rightarrow Y$ there are $m > n^*$ and a $\mathcal{K}$-arrow $g : Y \rightarrow U_m$ with $g \circ f \circ u_{n^*}^n = u_m^m$. 

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W.Kubiš (http://www.math.cas.cz/kubis/)
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\[
\cdots \to U_n \to U_{n^*} \to U_m \to \cdots
\]

\( f \)

\( g \)

\( Y \)
Theorem

Assume $\bar{u}$ is a weak Fraïssé sequence in $\mathcal{K}$ and $U = \lim \bar{u}$. Then $U$ is $\mathcal{K}$-generic.
Theorem

Assume $\vec{u}$ is a weak Fraïssé sequence in $\mathcal{K}$ and $U = \lim \vec{u}$. Then $U$ is $\mathcal{K}$-generic.

Proof.

\[ \cdots \rightarrow U_{n_0} \rightarrow U_{n_0}^* \rightarrow U_{n_1} \rightarrow U_{n_1}^* \rightarrow U_{n_2} \rightarrow \cdots \]

\[ A_0 \rightarrow A_2 \rightarrow A_4 \rightarrow A_6 \]
Weakenings of amalgamation

**Definition**

We say that $\mathcal{K}$ has the **cofinal amalgamation property (CAP)** if for every $Z \in \text{Obj}(\mathcal{K})$ there is a $\mathcal{K}$-arrow $e : Z \to Z'$ such that $\mathcal{K}$ has amalgamations at $Z'$. 

---

Definition (Ivanov 1999; Kechris & Rosendal 2007; Kruckman 2016) 

We say that $\mathcal{K}$ has the **weak amalgamation property (WAP)** if for every $Z \in \text{Obj}(\mathcal{K})$ there is a $\mathcal{K}$-arrow $e : Z \to Z'$ such that for every $\mathcal{K}$-arrows $f : Z' \to X$, $g : Z' \to Y$ there exist $\mathcal{K}$-arrows $f' : X \to W$, $g' : Y \to W$ such that $f' \circ e = g' \circ e$. 

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Proposition

Finite graphs of vertex degree \( \leq 2 \) have the CAP.
Proposition

*Finite graphs of vertex degree $\leq 2$ have the CAP.*
Weak injectivity

Definition

An object \( V \in \text{Obj}(\mathcal{L}) \) is **weakly \( \mathcal{K} \)-injective** if

- every \( \mathcal{K} \)-object has an \( \mathcal{L} \)-arrow into \( V \), and
- for every \( \mathcal{L} \)-arrow \( e: A \to V \) there exists a \( \mathcal{K} \)-arrow \( i: A \to B \) such that for every \( \mathcal{K} \)-arrow \( f: B \to Y \) there is an \( \mathcal{L} \)-arrow \( g: Y \to V \) satisfying \( g \circ f \circ i = e \).
Weak injectivity

**Definition**

An object $V \in \text{Obj}(\mathcal{L})$ is **weakly $\mathcal{K}$-injective** if

- every $\mathcal{K}$-object has an $\mathcal{L}$-arrow into $V$, and
- for every $\mathcal{L}$-arrow $e: A \to V$ there exists a $\mathcal{K}$-arrow $i: A \to B$ such that for every $\mathcal{K}$-arrow $f: B \to Y$ there is an $\mathcal{L}$-arrow $g: Y \to V$ satisfying $g \circ f \circ i = e$.

![Diagram](https://via.placeholder.com/150)

$A \xrightarrow{i} B \xrightarrow{f} Y$

$e \downarrow \quad g$

$V \leftarrow$
Theorem (Krawczyk & K. 2016)

Let $\mathcal{K}$ be a countable directed category of finitely generated models with embeddings.

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Let $\mathcal{K}$ be as above and let $U$ be a countably generated model. The following properties are equivalent:

(a) $U$ is $\mathcal{K}$-generic.

(b) Eve does not have a winning strategy in $BM(\mathcal{K}, U)$.

(c) $U$ is weakly $\mathcal{K}$-injective.
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(c) $U$ is weakly $\mathcal{K}$-injective.
The first example of a weak Fraïssé class with no CAP
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The first example of a weak Fraïssé class with no CAP


A quote from Pabion’s paper:

3° M. Pouzet m’a communiqué l’exemple suivant de relation uniformément prêhomogène et non pseudo-homogène. Sur Q, définir R (x, y, z) par x < y, x < z et y ≠ z.

(*) Séance du 7 février 1972.

Université Claude Bernard,
Mathématiques,
43, boulevard du Onze-Novembre 1918,
69-Villeurbanne, Rhône.
Weak Fraïssé theory

Definition

A **weak Fraïssé class** is a class $\mathcal{F}$ of finitely generated models of a fixed countable signature, closed under isomorphisms, having with many types, satisfying (JEP) and (WAP).
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## Theorem
Let $\mathcal{F}$ be a weak Fraïssé class. Then there exists a unique countable model $U$ that is weakly $\mathcal{F}$-injective and weakly homogeneous. Furthermore, $U$ is $\mathcal{F}$-generic.
Weak Fraïssé theory

**Definition**

A **weak Fraïssé class** is a class $\mathcal{F}$ of finitely generated models of a fixed countable signature, closed under isomorphisms, having with many types, satisfying (JEP) and (WAP).

**Theorem**

Let $\mathcal{F}$ be a weak Fraïssé class. Then there exists a unique countable model $U$ that is weakly $\mathcal{F}$-injective and weakly homogeneous. Furthermore, $U$ is $\mathcal{F}$-generic.

Conversely, given a countable weakly homogeneous model $M$, its age

$$\mathcal{F} = \{ A : A \text{ is finitely generated and embeddable into } M \}$$

is a weak Fraïssé class.
A structure $M$ is **weakly homogeneous** if for every finitely generated substructure $A \subseteq M$ there is a bigger finitely generated substructure $B \subseteq M$ containing $A$ such that every embedding $e : A \to M$ extendable to $B$ extends to an automorphism of $M$. 
Definition

A structure $M$ is **weakly homogeneous** if for every finitely generated substructure $A \subseteq M$ there is a bigger finitely generated substructure $B \subseteq M$ containing $A$ such that every embedding $e: A \rightarrow M$ extendable to $B$ extends to an automorphism of $M$. 
Some references

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Theorem (Krawczyk, Kruckman, Panagiotopoulos, K. 2018)

There exist continuum many hereditary weak Fraïssé classes of finite graphs without the cofinal AP.

Example

Let $G$ be the class of all finite acyclic graphs in which no two vertices of degree $>2$ are adjacent.

Then $G$ is a weak Fraïssé class failing the CAP.

W.Kubiš (http://www.math.cas.cz/kubis/)
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Example

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Theorem (Bartoš, K. 2018)

Let $\mathcal{K}$ be a class of non-degenerate Peano continua treated as a category with all continuous surjections. Then the pseudo-arc is $\mathcal{K}^{\text{op}}$-generic.

($\mathcal{K}^{\text{op}}$ is the category opposite to $\mathcal{K}$.)
Theorem (Bartoš, K. 2018)

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Theorem (Kwiatkowska, K. 2017)

The Poulsen simplex is generic over the (opposite) category of finite-dimensional simplices with affine surjections.
Let $M$ be a countable homogeneous structure. Is it always true that the group $\text{Aut}(M)$ contains isomorphic copies of all groups of the form $\text{Aut}(X)$, where $X$ is a substructure of $M$?
Let $M$ be a countable homogeneous structure. Is it always true that the group $\text{Aut}(M)$ contains isomorphic copies of all groups of the form $\text{Aut}(X)$, where $X$ is a substructure of $M$?

If this is the case, we shall say that $\text{Aut}(M)$ is universal.
Definition (Kuzeljević, K. 2018)

A structure $M$ is **uniformly homogeneous** if

1. $M$ is homogeneous and

2. for every finite substructure $A \subseteq M$ there exists an extension operator $e_A : \text{Aut}(A) \rightarrow \text{Aut}(M)$ such that

$$e_A(g \circ h) = e_A(g) \circ e_A(h)$$

for every $g, h \in \text{Aut}(A)$. 

Uniform homogeneity
A homogeneous digraph that is not uniformly homogeneous
A homogeneous digraph that is not uniformly homogeneous
Katětov functors

**Definition**

Let $\mathcal{F}$ be a class of finite structures of the same type and let $M$ be a countable homogeneous structure such that every $A \in \mathcal{F}$ embeds into $M$ and every finite substructure of $M$ is isomorphic to some $A \in \mathcal{F}$. A **Katětov functor** is a pair $\langle K, \eta \rangle$ such that $K$ assigns to each embedding $e: A \to B$ with $A, B \in \mathcal{F}$ an embedding $K(e): M \to M$, $\eta$ assigns to each $A \in \mathcal{F}$ an embedding $\eta_A: A \to M$. Furthermore, $K$ is a functor, i.e., $K(id_A) = id_M$, $K(e \circ f) = K(e) \circ K(f)$, and the following diagram commutes

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & M \\
e & \downarrow & \downarrow K(e) \\
B & \xrightarrow{\eta_B} & M
\end{array}
$$

for every embedding $e: A \to B$ with $A, B \in \mathcal{F}$.
Theorem (Mašulović & K.)

Assume $\langle \mathcal{F}, M \rangle$ admits a Katětov functor. Then for every substructure $X$ of $M$ there exists a topological group embedding

$$e_X : \text{Aut}(X) \to \text{Aut}(M).$$
Theorem (Mašulović & K.)

Assume $\langle F, M \rangle$ admits a Katětov functor. Then for every substructure $X$ of $M$ there exists a topological group embedding

$$e_X : \text{Aut}(X) \rightarrow \text{Aut}(M).$$

Claim (☹)

Most of the well known homogeneous relational structures admit a Katětov functor.
Theorem (Shelah & K. 2018)

There exists a countable homogeneous relational structure \( E \) such that:

- every finite group embeds into \( \text{Aut}(E) \),
- \( S_\infty \) does not embed into \( \text{Aut}(E) \),
- \( S_\infty \approx \text{Aut}(X) \) for some \( X \subseteq E \).

Furthermore, \( E \) is not uniformly homogeneous.
Theorem (Shelah & K. 2018)

There exists a countable homogeneous relational structure $M$ such that:

- $\text{Aut}(M)$ is torsion-free,
- for every $n \in \mathbb{N}$ there is a finite $A \subseteq M$ with $S_n \cong \text{Aut}(A)$.
Some more references


