Generic structures

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- a partial associative composition operation $\circ$ defined on arrows, where $f \circ g$ is defined $\iff$ the domain of g coincides with the domain of f.

Furthermore, for each $A \in \mathrm{Obj}(\mathfrak{K})$ there is an *identity* $\mathrm{id}_A \in \mathfrak{K}(A,A)$ satisfying $\mathrm{id}_A \circ g = g$ and $f \circ \mathrm{id}_A = f$ for $f \in \mathfrak{K}(A,X)$, $g \in \mathfrak{K}(Y,A)$, $X, Y \in \mathrm{Obj}(\mathfrak{K})$.

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Let $\vec{x}$ be a sequence in $\mathfrak{K}$. The colimit of $\vec{x}$ is a pair $\langle X, \{x_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $x_n^\infty \colon X_n \to X$ satisfying:

1. $x_n^\infty = x_m^\infty \circ x_n^m$ for every $n < m$.
2. If $\langle Y, \{y_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $y_n^\infty \colon X_n \to Y$ satisfies $y_n^\infty = y_m^\infty \circ y_n^m$ for every $n < m$ then there is a unique arrow $f \colon X \to Y$ satisfying $f \circ x_n^\infty = y_n^\infty$ for every $n \in \mathbb{N}$.

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More generally, after Odd's move finishing with an object A_{2k-1}, Eve chooses $A_{2k} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k-1}^{2k} \colon A_{2k-1} \to A_{2k}$.

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Next, Odd chooses $A_{2k+1} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k}^{2k+1} \colon A_{2k} \to A_{2k+1}$. And so on...

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The result of a play is a sequence $\vec{a}$:

$$A_0 \xrightarrow{a_0^1} A_1 \longrightarrow \cdots \longrightarrow A_{2k-1} \xrightarrow{a_{2k-1}^{2k}} A_{2k} \longrightarrow \cdots$$

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We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-generic if Odd has a strategy in the Banach-Mazur game BM $(\mathfrak{K})$ such that the colimit of the resulting sequence $\vec{a}$ is always isomorphic to U, no matter how Eve plays.

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Proof.

The rules for Eve and Odd are the same. □

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Example

Let $\mathfrak{K}$ be the category of all finite acyclic graphs with embeddings.
Then the countable everywhere infinitely branching tree is $\mathfrak{K}$-generic.

Theorem (Urysohn, 1927)

There exists a unique Polish metric space $\mathbb{U}$ *with the following property:*

(E) *For every finite metric spaces* $A \subseteq B$, *every isometric embedding* $e \colon A \to \mathbb{U}$ *can be extended to an isometric embedding* $f \colon B \to \mathbb{U}$.

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Theorem

Let $\mathfrak{M}_{\mathrm{fin}}$ be the category of finite metric spaces with isometric embeddings.
Then the Urysohn space $\mathbb{U}$ is $\mathfrak{M}_{\mathrm{fin}}$-generic.

The amalgamation property

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Definition

We say that $\mathfrak{K}$ has amalgamations at $Z \in \mathrm{Obj}(\mathfrak{K})$ if for every $\mathfrak{K}$-arrows $f\colon Z \to X$, $g\colon Z \to Y$ there exist $\mathfrak{K}$-arrows $f'\colon X \to W$, $g'\colon Y \to W$ such that $f' \circ f = g' \circ g$.

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We say that $\mathfrak{K}$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \mathrm{Obj}(\mathfrak{K})$.

Theorem (Universality)

Assume $\mathfrak{K}$ has the AP and U is $\mathfrak{K}$-generic.
Then for every $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{K}$, there exists an arrow

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Example

Let $\mathfrak{K}$ be the category of all finite linear graphs with embeddings. Then $\langle \mathbb{Z}, E \rangle$ is $\mathfrak{K}$-generic, where $xEy \iff |x - y| = 1$.
On the other hand, $\langle \mathbb{Z}, E \rangle \oplus \langle \mathbb{Z}, E \rangle \not\to \langle \mathbb{Z}, E \rangle$.

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- For every $n \in \omega$, for every $\mathfrak{K}$-arrow $f\colon U_n \to Y$ there are $m > n$ and a $\mathfrak{K}$-arrow $g\colon Y \to U_m$ such that $g \circ f = u_n^m$.

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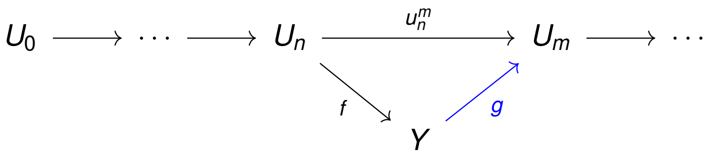

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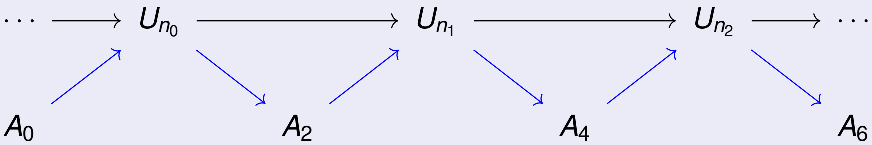

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1. For every $X, Y \in \mathrm{Obj}(\mathfrak{K})$ there is $U \in \mathrm{Obj}(\mathfrak{K})$ such that

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2. $\mathfrak{K}$ has the amalgamation property.

Theorem 2

Assume $\mathfrak{K} \subseteq \mathfrak{L}$ is such that every sequence in $\mathfrak{K}$ converges in $\mathfrak{L}$ and $\mathfrak{K}$ is a Fraïssé category. Then there exists a $\mathfrak{K}$-generic object in $\mathfrak{L}$.

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Let $\mathbb{P}$ be the poset of all finite sequences in $\mathfrak{K}$, i.e., covariant functors from some $n \in \omega$ into $\mathfrak{K}$. The ordering is end-extension.

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Let

$$\mathscr{D} = \{D_{n,f} \colon n \in \omega,\ f \in \mathfrak{K}\} \cup \{E_{n,A} \colon n \in \omega,\ X \in \operatorname{Obj}(\mathfrak{K})\},$$

where

$$D_{n,f} = \{\vec{x} \in \mathbb{P} \colon X_n = \operatorname{dom}(f) \implies (\exists m > n)(\exists g)\ g \circ f = x_n^m\},$$

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Let $\vec{u}$ be the sequence coming from a $\mathscr{D}$-generic filter/ideal. Then $\vec{u}$ is Fraïssé, therefore $U = \lim \vec{u}$ is $\mathfrak{K}$-generic. □

Fraïssé theory

Definition

A Fraïssé class is a class of finite models of a fixed countable language satisfying:

(H) For every $A \in \mathscr{F}$, every model isomorphic to a submodel of A is in $\mathscr{F}$.

(JEP) Every two models from $\mathscr{F}$ embed into a single model from $\mathscr{F}$.

(AP) $\mathscr{F}$ has the amalgamation property for embeddings.

(CMT) $\mathscr{F}$ has countably many isomorphic types.

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Conversely, if U is a countable homogeneous model then the class of all models isomorphic to finite submodels of U is Fraïssé.

More examples

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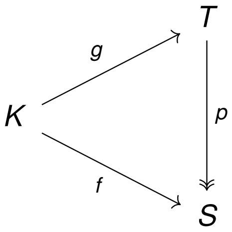

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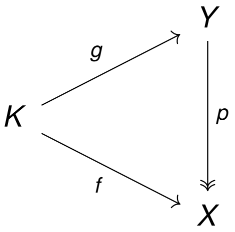

Theorem (Bielas, Walczyńska, K.)

Let 2^ω denote the Cantor set. A continuous mapping $\eta\colon K \to 2^\omega$ is $\mathfrak{K}_K$-generic $\iff \eta$ is a topological embedding and $\eta[K]$ is nowhere dense in 2^ω.

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Corollary (Knaster & Reichbach 1953)

Let $h\colon A \to B$ be a homeomorphism between closed nowhere dense subsets of 2^ω. Then there exists a homeomorphism $H\colon 2^\omega \to 2^\omega$ such that

$$H \restriction A = h.$$

The Gurarii space

Theorem (Gurarii 1966)

There exists a separable Banach space $\mathbb{G}$ with the following property.

(G) *For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every linear isometric embedding $e \colon E \to \mathbb{G}$ there exists a linear ε-isometric embedding $f \colon F \to \mathbb{G}$ such that $f \restriction E = e$.*

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Elementary proof: Solecki & K. 2013.

Theorem

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Key Lemma (Solecki & K.)

Let X, Y be finite-dimensional normed spaces, let $f \colon X \to Y$ be an ε-isometry with $0 < \varepsilon < 1$. Then there exist a finite-dimensional normed space Z and isometric embeddings $i \colon X \to Z$, $j \colon Y \to Z$ such that

$$\|i - j \circ f\| \leqslant \varepsilon.$$

The pseudo-arc

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