Generic structures

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A *category* $\mathcal{K}$ consists of

- a class of objects $\text{Obj}(\mathcal{K})$,
Categories

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- a class of arrows $\bigcup_{A,B\in\text{Obj}(\mathcal{K})} \mathcal{K}(A, B)$, where $f \in \mathcal{K}(A, B)$ means $A$ is the domain of $f$ and $B$ is the codomain of $f$,
- a partial associative composition operation $\circ$ defined on arrows, where $f \circ g$ is defined $\iff$ the domain of $g$ coincides with the domain of $f$. 

Furthermore, for each $A \in \text{Obj}(\mathcal{K})$ there is an identity $\text{id}_A \in \mathcal{K}(A, A)$ satisfying $\text{id}_A \circ g = g$ and $f \circ \text{id}_A = f$ for $f \in \mathcal{K}(A, X)$, $g \in \mathcal{K}(Y, A)$, $X, Y \in \text{Obj}(\mathcal{K})$. 

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A *category* \( \mathbb{K} \) consists of

- a class of objects \( \text{Obj}(\mathbb{K}) \),
- a class of arrows \( \bigcup_{A,B \in \text{Obj}(\mathbb{K})} \mathbb{K}(A, B) \), where \( f \in \mathbb{K}(A, B) \) means \( A \) is the *domain* of \( f \) and \( B \) is the *codomain* of \( f \),
- a partial associative composition operation \( \circ \) defined on arrows, where \( f \circ g \) is defined \( \iff \) the domain of \( g \) coincides with the domain of \( f \).

Furthermore, for each \( A \in \text{Obj}(\mathbb{K}) \) there is an *identity* \( \text{id}_A \in \mathbb{K}(A, A) \) satisfying \( \text{id}_A \circ g = g \) and \( f \circ \text{id}_A = f \) for \( f \in \mathbb{K}(A, X) \), \( g \in \mathbb{K}(Y, A) \), \( X, Y \in \text{Obj}(\mathbb{K}) \).
Definition

A sequence in $\mathcal{K}$ is a functor $\bar{x}$ from $\omega$ into $\mathcal{K}$. 

Definition

Let $\bar{x}$ be a sequence in $\mathcal{K}$. The colimit of $\bar{x}$ is a pair $\langle X, \{x_\infty^n\}_{n \in \mathbb{N}} \rangle$ with $x_\infty^n : X_n \to X$ satisfying:

1. $x_\infty^n = x_\infty^m \circ x_m^n$ for every $n < m$.

2. If $\langle Y, \{y_\infty^n\}_{n \in \mathbb{N}} \rangle$ with $y_\infty^n : X_n \to Y$ satisfies $y_\infty^n = y_\infty^m \circ y_m^n$ for every $n < m$ then there is a unique arrow $f : X \to Y$ satisfying $f \circ x_\infty^n = y_\infty^n$ for every $n \in \mathbb{N}$. 

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$$X_0 \xrightarrow{x_0^1} X_1 \xrightarrow{x_1^2} X_2 \xrightarrow{x_2^3} \ldots$$
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Definition

Let $\vec{x}$ be a sequence in $\mathcal{K}$. The colimit of $\vec{x}$ is a pair $\langle X, \{x_n^\infty\}_{n \in \mathbb{N}} \rangle$ with $x_n^\infty : X_n \to X$ satisfying:

1. $x_n^\infty = x_m^\infty \circ x_n^m$ for every $n < m$.
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The Banach-Mazur game

Definition

The Banach-Mazur game $BM(K)$ played on $K$ is described as follows.

There are two players: Eve and Odd.

Eve starts by choosing $A_0 \in \text{Obj}(K)$.

Then Odd chooses $A_1 \in \text{Obj}(K)$ together with a $K$-arrow $a_{01} : A_0 \to A_1$.

More generally, after Odd's move finishing with an object $A_{2k-1}$, Eve chooses $A_{2k} \in \text{Obj}(K)$ together with a $K$-arrow $a_{2k-1,2k} : A_{2k-1} \to A_{2k}$.

Next, Odd chooses $A_{2k+1} \in \text{Obj}(K)$ together with a $K$-arrow $a_{2k+1,2k+2} : A_{2k+1} \to A_{2k+2}$. And so on...

The result of a play is a sequence $\vec{a} : A_0 \xrightarrow{a_{01}} A_1 \xrightarrow{a_{2k-1,2k}} A_{2k} \xrightarrow{a_{2k+1,2k+2}} A_{2k+2} \cdots$. 
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The result of a play is a sequence $\vec{a}$:

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{} \cdots \xrightarrow{} A_{2k-1} \xrightarrow{a_{2k}} A_{2k} \xrightarrow{} \cdots$$
Generic objects

**General assumption:** \( K \subseteq L \).

**Definition**
We say that \( U \in \text{Obj}(L) \) is \( K \)-generic if Odd has a strategy in the Banach-Mazur game \( BM(K) \) such that the colimit of the resulting sequence \( \vec{a} \) is always isomorphic to \( U \), no matter how Eve plays.

**Proposition**
A \( K \)-generic object, if exists, is unique up to isomorphism.

**Proof.**
The rules for Eve and Odd are the same.
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We say that $U \in \text{Obj}(\mathcal{L})$ is $\mathcal{K}$-generic if Odd has a strategy in the Banach-Mazur game $\text{BM}(\mathcal{K})$ such that the colimit of the resulting sequence $\vec{a}$ is always isomorphic to $U$, no matter how Eve plays.

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General assumption: \( k \subseteq l \).

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A \( k \)-generic object, if exists, is unique up to isomorphism.

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The rules for Eve and Odd are the same.
Example

Let $\mathcal{K}$ be the category of all finite linearly ordered sets with embeddings. Then $\langle \mathbb{Q}, < \rangle$ is $\mathcal{K}$-generic.

Example

Let $\mathcal{K}$ be the category of all finite graphs with embeddings. Then the Rado graph $R = \langle \mathbb{N}, E_R \rangle$ is $\mathcal{K}$-generic, where $k < n$ are adjacent if and only if the $k$th digit in the binary expansion of $n$ is one.

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Let $\mathcal{K}$ be the category of all finite acyclic graphs with embeddings. Then the countable everywhere infinitely branching tree is $\mathcal{K}$-generic.
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Example
Let $\mathcal{K}$ be the category of all finite acyclic graphs with embeddings. Then the countable everywhere infinitely branching tree is $\mathcal{K}$-generic.
Theorem (Urysohn, 1927)

There exists a unique Polish metric space $U$ with the following property:

(E) For every finite metric spaces $A \subseteq B$, every isometric embedding $e: A \to U$ can be extended to an isometric embedding $f: B \to U$. 
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There exists a unique Polish metric space $\mathbb{U}$ with the following property:

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Furthermore:

- Every separable metric space embeds into $\mathbb{U}$.
- Every isometry between finite subsets of $\mathbb{U}$ extends to a bijective isometry of $\mathbb{U}$.
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Furthermore:

- Every separable metric space embeds into $\mathbb{U}$.
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Theorem

Let $\mathcal{M}_{\text{fin}}$ be the category of finite metric spaces with isometric embeddings.

Then the Urysohn space $\mathbb{U}$ is $\mathcal{M}_{\text{fin}}$-generic.
The amalgamation property

Definition

We say that $K$ has amalgamations at $Z \in \text{Obj}(K)$ if for every $K$-arrows $f: Z \to X$, $g: Z \to Y$ there exist $K$-arrows $f': X \to W$, $g': Y \to W$ such that $f' \circ f = g' \circ g$.

We say that $K$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \text{Obj}(K)$.
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\[ \begin{array}{ccc}
Y & \xrightarrow{g} & W \\
\uparrow{g} & & \uparrow{f'} \\
Z & \xrightarrow{f} & X \\
\end{array} \]
The amalgamation property

Definition

We say that $\mathcal{R}$ has **amalgamations at** $Z \in \text{Obj}(\mathcal{R})$ if for every $\mathcal{R}$-arrows $f : Z \to X$, $g : Z \to Y$ there exist $\mathcal{R}$-arrows $f' : X \to W$, $g' : Y \to W$ such that $f' \circ f = g' \circ g$.

We say that $\mathcal{R}$ has the **amalgamation property (AP)** if it has amalgamations at every $Z \in \text{Obj}(\mathcal{R})$. 

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Theorem (Universality)

Assume $K$ has the AP and $U$ is $K$-generic. Then for every $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $K$, there exists an arrow

$$e : X \to U.$$
**Theorem (Universality)**

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**Example**

Let $\mathcal{K}$ be the category of all finite linear graphs with embeddings. Then $\langle \mathbb{Z}, E \rangle$ is $\mathcal{K}$-generic, where $xEy \iff |x - y| = 1$. On the other hand, $\langle \mathbb{Z}, E \rangle \oplus \langle \mathbb{Z}, E \rangle \nleftrightarrow \langle \mathbb{Z}, E \rangle$. 
Fraïssé sequences

Definition

A Fraïssé sequence in $\mathcal{K}$ is a sequence $\vec{u}: \omega \to \mathcal{K}$ satisfying the following conditions:

1. For every $A \in \text{Obj}(\mathcal{K})$ there is $n$ such that $\mathcal{K}(A, U_n) \neq \emptyset$.
2. For every $n \in \omega$, for every $\mathcal{K}$-arrow $f: U_n \to Y$ there are $m > n$ and a $\mathcal{K}$-arrow $g: Y \to U_m$ such that $g \circ f = u_{mn}$.
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Diagram:

$$
\begin{array}{ccccccc}
U_0 & \longrightarrow & \cdots & \longrightarrow & U_n & \longrightarrow & u_n^m & \longrightarrow & U_m & \longrightarrow & \cdots \\
& & & & f & & g & & \\
& & Y & & & & \\
\end{array}
$$
Theorem 1

Let $\bar{u}$ be a Fraïssé sequence in $\mathfrak{H}$ and let $U = \lim \bar{u}$. Then $U$ is $\mathfrak{H}$-generic.
Theorem 1

Let $\vec{u}$ be a Fraïssé sequence in $\mathcal{K}$ and let $U = \lim \vec{u}$. Then $U$ is $\mathcal{K}$-generic.

Proof.

\[ \cdots \rightarrow U_{n_0} \rightarrow U_{n_1} \rightarrow U_{n_2} \rightarrow \cdots \]

\[ A_0 \rightarrow A_2 \rightarrow A_4 \rightarrow A_6 \]
Fraïssé categories

Definition

A Fraïssé category is a countable category $\mathcal{K}$ satisfying:

1. For every $X, Y \in \text{Obj}(\mathcal{K})$ there is $U \in \text{Obj}(\mathcal{K})$ such that $\mathcal{K}(X, U) \neq \emptyset \neq \mathcal{K}(Y, U)$. 

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2. \( \mathcal{K} \) has the amalgamation property.
Theorem 2

Assume $\mathcal{R} \subseteq \mathcal{L}$ is such that every sequence in $\mathcal{R}$ converges in $\mathcal{L}$ and $\mathcal{R}$ is a Fraïssé category. Then there exists a $\mathcal{R}$-generic object in $\mathcal{L}$.
Theorem 2

Assume $\mathcal{K} \subseteq \mathcal{L}$ is such that every sequence in $\mathcal{K}$ converges in $\mathcal{L}$ and $\mathcal{K}$ is a Fraïssé category. Then there exists a $\mathcal{K}$-generic object in $\mathcal{L}$.

Proof.

Let $\mathcal{P}$ be the poset of all finite sequences in $\mathcal{K}$, i.e., covariant functors from some $n \in \omega$ into $\mathcal{K}$. The ordering is end-extension.
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Let

$$\mathcal{D} = \{D_{n,f}: n \in \omega, f \in \mathcal{K}\} \cup \{E_{n,A}: n \in \omega, X \in \text{Obj}(\mathcal{K})\},$$

where

$$D_{n,f} = \{\vec{x} \in \mathbb{P}: X_n = \text{dom}(f) \implies (\exists m > n)(\exists g) g \circ f = x^n_m\},$$

$$E_{n,A} = \{\vec{x} \in \mathbb{P}: (\exists m \geq n) \mathcal{K}(A, X_m) \neq \emptyset\}.$$
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$$E_{n,A} = \{ \bar{x} \in \mathbb{P} : (\exists m \geq n) \mathcal{R}(A, X_m) \neq \emptyset \}.$$

Let $\bar{u}$ be the sequence coming from a $\mathcal{D}$-generic filter/ideal. Then $\bar{u}$ is Fraïssé, therefore $U = \lim \bar{u}$ is $\mathcal{R}$-generic.
Definition

A **Fraïssé class** is a class of finite models of a fixed countable language satisfying:

(H) For every $A \in \mathcal{F}$, every model isomorphic to a submodel of $A$ is in $\mathcal{F}$.

(JEP) Every two models from $\mathcal{F}$ embed into a single model from $\mathcal{F}$.

(AP) $\mathcal{F}$ has the amalgamation property for embeddings.

(CMT) $\mathcal{F}$ has countably many isomorphic types.
Theorem (Fraïssé, 1954)

Let $\mathcal{F}$ be a Fraïssé class. Then there exists a unique, up to isomorphism, countable model $U$ such that

1. $\mathcal{F}$ consists of all isomorphic types of finite submodels of $U$,
2. every isomorphism of finite submodels of $U$ extends to an automorphism of $U$ (in other words, $U$ is ultra-homogeneous).

Conversely, if $U$ is a countable homogeneous model then the class of all models isomorphic to finite submodels of $U$ is Fraïssé.
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More examples
The Cantor set

Fix a compact 0-dimensional space $K$. Define the category $K$ as follows. The objects are continuous mappings $f: K \to S$ with $S$ finite. An arrow from $f: K \to S$ to $g: K \to T$ is a surjection $p: T \to S$ satisfying $p \circ g = f$. 

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![Diagram](image-url)
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Let $\mathcal{L}_K$ be the category whose objects are continuous mappings $f: K \to X$ with $X$ metrizable compact 0-dimensional. An $\mathcal{L}_K$-arrow from $f: K \to X$ to $g: K \to Y$ is a continuous surjection $p: Y \to X$ satisfying $p \circ g = f$.
Theorem (Bielas, Walczyńska, K.)

Let $2^\omega$ denote the Cantor set. A continuous mapping $\eta: K \to 2^\omega$ is $K$-generic $\iff$ $\eta$ is a topological embedding and $\eta[K]$ is nowhere dense in $2^\omega$. 

Corollary (Knaster & Reichbach 1953)

Let $h: A \to B$ be a homeomorphism between closed nowhere dense subsets of $2^\omega$. Then there exists a homeomorphism $H: 2^\omega \to 2^\omega$ such that $H \upharpoonright A = h$. 
Theorem (Bielas, Walczyńska, K.)

Let $2^\omega$ denote the Cantor set. A continuous mapping $\eta: K \to 2^\omega$ is $\mathbb{K}$-generic $\iff$ $\eta$ is a topological embedding and $\eta[K]$ is nowhere dense in $2^\omega$.

Corollary (Knaster & Reichbach 1953)

Let $h: A \to B$ be a homeomorphism between closed nowhere dense subsets of $2^\omega$. Then there exists a homeomorphism $H: 2^\omega \to 2^\omega$ such that

$$H \upharpoonright A = h.$$
The Gurarii space

Theorem (Gurarii 1966)

There exists a separable Banach space $G$ with the following property.

$\text{(G)}$ For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every linear isometric embedding $e: E \to G$ there exists a linear $\varepsilon$-isometric embedding $f: F \to G$ such that $f \upharpoonright E = e$. 

Theorem (Lusky 1976)

Among separable spaces, property (G) determines the space $G$ uniquely up to linear isometries.

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There exists a separable Banach space $\mathbb{G}$ with the following property.

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Theorem

The Gurarii space $G$ is generic over the category $\mathcal{B}_{fd}$ of finite-dimensional normed spaces with linear isometric embeddings.
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Key Lemma (Solecki & K.)

Let $X$, $Y$ be finite-dimensional normed spaces, let $f : X \to Y$ be an $\varepsilon$-isometry with $0 < \varepsilon < 1$. Then there exist a finite-dimensional normed space $Z$ and isometric embeddings $i : X \to Z$, $j : Y \to Z$ such that

$$\|i - j \circ f\| \leq \varepsilon.$$
Let \( \mathcal{I} \) be the category of all continuous surjections from the unit interval \([0, 1]\) onto itself.
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The pseudo-arc

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Theorem

*The pseudo-arc is $\mathcal{I}$-generic.*