

Singular cardinals and compactness

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Plan of the lectures:

- Preamble: Singular cardinals, compactness
- Singular compactness theorem
- Constructions of non-compact objects
- Consistency results

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- Reflection/compactness phenomena such as stationary reflection behave differently: for example if κ is regular then κ^+ has a non-reflecting stationary subset, but this is false in general for singular κ .

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- The value of the continuum function at a singular strong limit cardinal κ is closely tied to its values below, but for κ regular we can use the Cohen poset $\text{Add}(\kappa, \lambda)$ to show this is not the case.
- Reflection/compactness phenomena such as stationary reflection behave differently: for example if κ is regular then κ^+ has a non-reflecting stationary subset, but this is false in general for singular κ .
- Consistency and independence results involving singular cardinals and their successors tend to be harder and involve larger cardinals than parallel results for other cardinals.

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- In the absence of large cardinals, there are inner models of V called “core models” which have L -like combinatorics (square, diamond, GCH) and which compute the successors of V -singulars correctly.
- On a more positive note, the fact that a singular cardinal κ is the union of fewer than κ sets of size less than κ powers types of combinatorial argument that are not available at regular cardinals. PCF theory is a salient example.

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- (Combinatorial set theory) A stationary subset S of a regular uncountable cardinal κ *reflects* if and only if there is $\alpha < \kappa$ such that $\text{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α .

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- (Cardinal arithmetic) Silver's theorem asserts that if GCH fails at a singular strong limit cardinal κ of uncountable cofinality, then it fails for almost every $\mu < \kappa$.

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These are both true for λ measurable, both false for (eg)
 $\lambda = \aleph_1$.

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- A *transversal* for a family of non-empty sets is a 1-1 choice function.

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- There's a substructure 0 which is minimal under inclusion.
- For any two substructures A, B there is a unique minimal substructure $A + B$ which contains $A \cup B$.
- The union of a continuous chain of substructures is a substructure.

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The crucial idea is to relativise the notion of freeness, that is to introduce a notion “ B is free over A ” where A is a substructure of B . The intention is that the free structures should be the ones which are free over 0. Typically the definition of B 's being free over A will imply that any witness for A extends to a witness for B .

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- When A, B are non-empty families of countable sets and $A \subseteq B$, then B is free over A iff $B \setminus A$ has a transversal (say g) which takes values outside $\bigcup A$.

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Note: It's often true (and is true in our two running examples) that if C/A is free, then B/A for all B intermediate between A and C . But we don't need this.

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II B_0 B_1

The rules are that $B_0 = 0$, A_i and (for $i > 0$) B_i have size κ , $B_0 \subseteq A_0 \subseteq B_1 \subseteq \dots$ and B_{n+1}/B_n is free for all n . The first player to violate the rules loses, if the rules are followed forever then *II* wins.

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The rules are that C_i, D_i are structures of size κ and that $C_0 \subseteq D_0 \subseteq C_1 \dots$. The first player to violate these rules loses, and if the rules are never violated then II wins iff $B' + D_\omega$ is free over $B + D_\omega$, where $D_\omega = \bigcup_{n < \omega} D_n = \bigcup_{n < \omega} C_n$.

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We also fix σ_i which is winning for II in $G_1(\lambda_i)$ when $i < \mu$ is not a limit ordinal, and $\tau_i(B, B')$ which is winning in $G_2(\lambda_i, \lambda_{i+1}, B, B')$ for all $i < \mu$ and all relevant B, B' .

Construction: we will build a matrix of substructures with ω rows and μ columns:

$$\begin{array}{cccc}
 \vdots & \vdots & \vdots & \ddots \\
 A_1^0 & A_1^1 & A_1^2 & \dots \\
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where:

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- Column i is increasing for all $i < \mu$. As a consequence, $\bigcup_{n < \omega} A_n^i = \bigcup_{n < \omega} B_n^i$. We will denote this structure by B_ω^i .

- The “ n^{th} A -row” $(A_n^i)_{i < \mu}$ is increasing and continuous for all $n < \omega$. As a consequence, the sequence $(B_\omega^i)_{i < n}$ is increasing and continuous with union \mathcal{M} .

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- For non-limit i , column i is a run of the game $G_1(\lambda_i)$ where player II is playing the structures B_n^i according to the winning strategy σ_i . As a consequence, B_{n+1}^i/B_n^i is free for all $n < \omega$.

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- For every $i < \mu$ and every pair B_n^{i+1}, B_{n+1}^{i+1} in column $i + 1$, there is a run of the game $G_2(\lambda_i, \lambda_{i+1}, B_n^{i+1}, B_{n+1}^{i+1})$

$$\begin{array}{rcc}
 \text{I} & B_{n+1}^i & B_{n+2}^i \\
 & & \dots \\
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 \end{array}$$

where II is playing according to the winning strategy $\tau_i(B_n^{i+1}, B_{n+1}^{i+1})$. As a consequence, $(B_{n+1}^{i+1} + B_\omega^i)/(B_n^{i+1} + B_\omega^i)$ is free.

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- For each limit $j < \mu$, $\bigcup_{i < j, n < \omega} D_{\omega \cdot i+n} = B_\omega^j = D_{\omega \cdot j}$.

How do we do it? The main issue is that we need the A -rows to be continuously increasing, and we need every column (including limit columns) to be constructed according to strategies for the game G_2 . It is here that λ being singular (in particular $\mu < \lambda_0$) will be crucial.

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We will build the matrix of sets row by row.

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- The first two rows are easy: $B_0^i = 0$ for all i , and $(A_0^i)_{i < \mu}$ is any increasing and continuous chain of substructures with $|A_0^i| = \lambda_i$ and $\bigcup_{i < \mu} A_0^i = \mathcal{M}$.

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- The “ B -rows” with positive subscripts are also easy: for non-limit i we compute B_{n+1}^i from A_0^i, \dots, A_n^i and the strategy σ_i , for limit i let $B_{n+1}^i = A_n^i$.

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$$\text{I} \quad B_{m+1}^i \quad \dots \quad B_{n+1}^i$$

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of the game $G_2(\lambda_i, \lambda_{i+1}, B_m^{i+1}, B_{m+1}^{i+1})$, where player II is playing according to the winning strategy $\tau_i(B_m^{i+1}, B_{m+1}^{i+1})$.

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Define an auxiliary set C_{n+1}^i such that $B_{n+1}^i \subseteq C_{n+1}^i$, $|C_{n+1}^i| = \lambda_i$, and $D_{n-m}^{i,m} \subseteq C_{n+1}^i$ for all $m \leq n$.

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- Since $\mu < \lambda_0 \leq \lambda_i$, we see that $|A_{n+1}^i| = \lambda_i$. The other key points are that $B_{n+1}^i \subseteq C_{n+1}^i \subseteq A_{n+1}^i$, and that $(A_{n+1}^i)_{i < \mu}$ is continuous and increasing with i .

This concludes the proof. As we see shortly, we will need λ to be a limit cardinal to see that we can win $G_1(\kappa)$ for $\kappa < \lambda$. We needed λ *singular* to do the “looking ahead to all subsequent columns” in the main construction.

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- If $(A_i)_{i < \kappa^+}$ is an increasing and continuous chain of structures of size κ such that $\bigcup_{i < \kappa^+} A_i/A_0$ is free, then there is a club set $C \subseteq \kappa^+$ such that $0 \in C$ and A_j/A_i is free for $i, j \in C$ with $i < j$.

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Key idea for the first game: If all (many) substructures of size κ^+ are free (that is free over 0) then II has a winning strategy for $G_1(\kappa)$. It's important that λ is a limit cardinal, since we want to win for unboundedly many $\kappa < \lambda$.

We appeal to the well-known Gale-Stewart theorem on the determinacy of open games. The game is closed for player II, so if II does not win then I wins with some strategy σ .

We fix some large regular θ and build a continuous increasing chain $(M_i)_{i < \kappa^+}$ of elementary substructures of H_θ such that:

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- $\mathcal{M}, \sigma \in M_0$.
- $\kappa + 1 \subseteq M_i, |M_i| = \kappa$ for all $i < \kappa^+$.
- $\langle M_i : i \leq j \rangle \in M_{j+1}$ for all $j < \kappa$.

Let $M_\infty = \bigcup_{i < \kappa^+} M_i$, so that by hypothesis $M_\infty \cap \mathcal{M}$ is free. By our added assumption about “free over”, we can find (taking the first ω points of an appropriate club) a strictly increasing ω -sequence (B_n) of structures of size κ such that $B_0 = 0$, $B_n = M_{\alpha_n} \cap \mathcal{M}$ for increasing $\alpha_n < \kappa^+$, and B_{n+1}/B_n is free for all n .

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I	$\sigma(B_0)$	$\sigma(B_0, B_1)$	
			...
II	B_0	B_1	

This is a legitimate run of the game because σ , \mathcal{M} and the models M_{α_i} for $0 < i \leq n$ are all elements of $M_{\alpha_{n+1}}$. So $\sigma(B_0, \dots, B_n) \in M_{\alpha_{n+1}}$, and hence easily $\sigma(B_0, \dots, B_n) \subseteq B_{n+1}$.

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Remark: It was an overkill to assume that *all* substructures of \mathcal{M} with size κ^+ are free.

Adding more axioms about freeness, Shelah proved a result about G_2 parallel to the one I proved for G_1 . But it turns out that in many cases (including my two running examples) we don't need any assumption about the ambient structure \mathcal{M} to prove that player II wins G_2 .

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Abelian groups: Let X be a set of coset representatives for a basis of B'/B . After I plays C_n , II finds $X_n \subseteq X$ of size κ such that every element of $C_n \cap B'$ is congruent mod B to something in $\text{span}(X_n)$, and then lets $D_n = C_n + \text{span}(X_n)$. Now check $X \setminus \bigcup_n X_n$ gives coset representatives for a basis of $(B' + D_\omega)/(B + D_\omega)$.

Transversals: Let g be a transversal of $B' \setminus B$ which does not take any value in $\bigcup B$. After I plays C_n , II finds $D_n \supseteq C_n$ such that $|D_n| = \kappa$ and $g(x) \in \bigcup C_n \implies x \in D_n$, this is possible because g is 1-1. Now check that $g \upharpoonright B' \setminus (B \cup D_\omega)$ does not take any value in $\bigcup D_\omega$.

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By the compactness theorem for first order logic, $PT(\lambda, \omega)$ holds for all λ .

Let μ and τ be regular cardinals with $\mu < \tau$, and let $S \subseteq \tau \cap \text{cof}(\mu)$. A *ladder system* on S is a sequence $(x_\delta)_{\delta \in S}$ such that x_δ is cofinal in δ with order type μ .

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By Fodor, there is no transversal. An easy diagonalisation shows that every countable subset has a transversal.

For τ regular and uncountable, a *non-reflecting stationary subset (NRSS)* of τ is $S \subseteq \tau$ such that S is stationary, and $S \cap \gamma$ is non-stationary for all $\gamma \in \tau \cap \text{cof}(> \omega)$.

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Key fact: If S is a NRSS of τ , and $(x_\delta)_{\delta \in S}$ is a ladder system then for all $\gamma < \tau$ we can choose disjoint tails of x_δ for $\delta \in S \cap \gamma$.

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Proof by induction on γ . If $\gamma = \gamma_0 + 1$ then nothing to do unless $\gamma_0 \in S$, in which case apply IH for $\delta < \gamma_0$ and then use $\sup(x_\delta \cap x_{\gamma_0}) < \delta$ to ensure disjointness from x_{γ_0} .

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Remark: By same argument, if $S \subseteq \omega_1$ is non-stationary then a ladder system on S has disjoint tails, in particular it has a transversal.

(Milner-Shelah) If $\kappa < \lambda$ regular and there is $S \subseteq \lambda \cap \text{cof}(\kappa)$ a NRSS of λ , then $NPT(\kappa, \omega_1)$ implies $NPT(\lambda, \omega_1)$.

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For any regular κ , $\kappa^+ \cap \text{cof}(\kappa)$ is NRSS of κ^+ . Since we know $NPT(\aleph_1, \aleph_1)$, deduce $NPT(\aleph_n, \aleph_1)$ for $1 \leq n < \omega$.

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But Magidor showed that modulo large cardinals (ω supercompact cardinals) that consistently every stationary subset of $\aleph_{\omega+1}$ reflects.

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- There exist unbounded $A \subseteq \omega$, and a sequence $(f_\alpha)_{\alpha < \aleph_{\omega+1}}$ which is increasing and cofinal in $(\prod_{n \in A} \aleph_n, <^*)$.
Adjusting A and f 's we may assume that $0 \notin A$ and $f_\alpha(n) \in [\aleph_n, \aleph_{n+1})$.

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Adjusting A and f 's we may assume that $0 \notin A$ and $f_\alpha(n) \in [\aleph_n, \aleph_{n+1})$.
- Let $\alpha < \aleph_{\omega+1}$ be a limit ordinal. An *exact upper bound* for $(f_\beta)_{\beta < \alpha}$ is $g \in \prod_{n \in A} \aleph_n$ such that $\{h \in \prod_{n \in A} \aleph_n : h <^* g\} = \{h \in \prod_{n \in A} \aleph_n : \exists \beta < \alpha h <^* f_\beta\}$.
If an eub exists it is unique mod finite.

- If $\text{cf}(\alpha) > \omega$ and α is a point where an eub g exists with $\text{cf}(g(n)) > \omega$ for all n , then $\text{cf}(g(n)) = \text{cf}(\alpha)$ for all large n . Such α are called *good*. α is good iff there are $I \subseteq \alpha$ unbounded and $m < \omega$ such that $(f_\beta(n))_{\beta \in I}$ is strict increasing for $m \leq n < \omega$.

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- There are stationarily many good points in each uncountable cofinality.

Let T be the stationary set of good points of cofinality \aleph_1 . PCF theory gives structural information about $T \cap \gamma$ for $\gamma < \aleph_{\omega+1}$ with $\omega_1 < \text{cf}(\gamma)$:

- If γ is good, then almost all points in $\gamma \cap \text{cof}(\omega_1)$ are in T .
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For the experts: If γ is good, fix I and n witnessing this: all α of cofinality ω_1 such that I is unbounded in α are good. If γ is ungood, it is in the Bad or Ugly cases of Shelah's trichotomy: in either case the witnessing objects witness ungoodness almost everywhere below.

Viewed as sets of ordered pairs, the f_α 's form an almost disjoint family of countable subsets of $A \times \aleph_\omega$. To emphasise that we are think of them as sets, we write $A_\alpha = \{(m, f_\alpha(m)) : \alpha \in A\}$. Ordering A_α by first entries, we have a notion of "tail of A_α ".

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Idea of proof: Construct a witness to $NPT(\aleph_{\omega+1}, \aleph_2)$ and then "step down" to get a witness to $NPT(\aleph_{\omega+1}, \aleph_1)$.

Key claim: for all $\gamma < \aleph_{\omega+1}$ there exist (B_α, D_α) for $\alpha \in T \cap \gamma$ such that:

- B_α is a tail of A_α .
- D_α is club in α with $\text{ot}(D_\alpha) = \omega_1$.
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Assuming key claim, we fix E_α club in α for each $\alpha \in T$ and claim that $\{A_\alpha \times E_\alpha : \alpha \in T\}$ exemplify $NPT(\kappa^+, \aleph_2)$. There is no transversal of the whole system (freeze 1st coordinate on a stationary set, then apply Fodor on 2nd coordinate). For $\gamma < \aleph_{\omega+1}$ apply the key claim and see that $B_\alpha \times (D_\alpha \cap E_\alpha) \subseteq A_\alpha \times E_\alpha$, these subsets are nonempty and pairwise disjoint.

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Easy case 2: There exist $\gamma_i \notin T$ increasing continuous and cofinal in γ . Use γ_i 's to cut γ into blocks, apply IH in each block, then replace D_α for $\alpha \in [\gamma_i, \gamma_{i+1})$ by tail above γ_i .

Hard case: None of the above. By assumption γ is good and $\text{cf}(\gamma) > \omega_1$. Fix $I \subseteq \gamma$ cofinal and m such that $(f_\alpha(n))_{\alpha \in I}$ is increasing for $n \geq m$.

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C is the club of $\alpha < \gamma$ such that I is unbounded in α :
decompose γ into $\lim(C)$ and points which live in an interval $(\delta, \eta]$ where δ, η are successive points of C .

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For $\alpha \in S \cap \lim(C)$ choose $D_\alpha = C \cap \alpha$, so that $D_\alpha \cap D_\beta = \emptyset$ for $\beta < \alpha$ unless also $\beta \in S \cap \lim(C)$.

Key point: Fix $\alpha \in S \cap \lim(C)$. For every $\beta \in I \cap \alpha$, there is $n(\beta) \geq m$ such that $f_\beta(n) < f_\alpha(n)$ for $n \geq n(\beta)$. As $\text{cf}(\alpha) = \omega_1$, there is $J \subseteq I \cap \alpha$ unbounded and n^* such that $n(\beta) = n^*$ for $\beta \in J$. But then (by choice of I and m) $f_\beta(n) < f_\alpha(n)$ for all $\beta \in I \cap \alpha$ and $n \geq n^*$.

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Let $\eta(\alpha)$ be the least point of I above α . Then we can choose $m(\alpha) \geq m$ such that for $n \geq m(\alpha)$:

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For $n \geq m(\alpha), m(\alpha')$,

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It follows that $B_\alpha \cap B_{\alpha'} = \emptyset$.

So far we just proved $NPT(\aleph_{\omega+1}, \aleph_2)$. To bring it down to $NPT(\aleph_{\omega+1}, \aleph_1)$, we fix a ladder system S_γ on the countable limit ordinals, and enumerate each E_α as $e_{\alpha,\gamma}$.

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No transversal? For every α there is γ such that $B_{\alpha,\gamma}$ maps to something chosen from the second coordinate, contradiction by $A_\alpha \subseteq A \times \aleph_\omega$ and Fodor.

Transversal for $\{B_{\alpha,\gamma} : \alpha < \eta, \gamma < \aleph_1\}$?

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No transversal? For every α there is γ such that $B_{\alpha,\gamma}$ maps to something chosen from the second coordinate, contradiction by $A_\alpha \subseteq A \times \aleph_\omega$ and Fodor.

Transversal for $\{B_{\alpha,\gamma} : \alpha < \eta, \gamma < \aleph_1\}$? Choose B_α and D_α for $\alpha < \eta$ such that $B_\alpha \times D_\alpha$'s are pairwise disjoint. Fix α . If $e_{\alpha,\gamma} \in D_\alpha$ then choose a point in $B_\alpha \times \{e_{\alpha,\gamma}\}$. On the non-stationary set of γ such that $e_{\alpha,\gamma} \notin D_\alpha$, choose a transversal of corresponding S_γ 's and use this to select a point in $S_\gamma \times \{\alpha\}$