Singular cardinals and compactness

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Plan of the lectures:

- Preamble: Singular cardinals, compactness
- Singular compactness theorem
- Constructions of non-compact objects
- Consistency results

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- Reflection/compactness phenomena such as stationary reflection behave differently: for example if κ is regular then κ^+ has a non-reflecting stationary subset, but this is false in general for singular κ.

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- The value of the continuum function at a singular strong limit cardinal κ is closely tied to its values below, but for κ regular we can use the Cohen poset $\mathrm{Add}(\kappa, \lambda)$ to show this is not the case.
- Reflection/compactness phenomena such as stationary reflection behave differently: for example if κ is regular then κ^+ has a non-reflecting stationary subset, but this is false in general for singular κ.
- Consistency and independence results involving singular cardinals and their successors tend to be harder and involve larger cardinals than parallel results for other cardinals.

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- In the absence of large cardinals, there are inner models of V called "core models" which have L-like combinatorics (square, diamond, GCH) and which compute the successors of V-singulars correctly.
- On a more positive note, the fact that a singular cardinal κ is the union of fewer than κ sets of size less than κ powers types of combinatorial argument that are not available at regular cardinals. PCF theory is a salient example.

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- (Combinatorial set theory) A stationary subset S of a regular uncountable cardinal κ *reflects* if and only if there is $\alpha < \kappa$ such that $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α.

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- (Combinatorial set theory) A stationary subset S of a regular uncountable cardinal κ *reflects* if and only if there is $\alpha < \kappa$ such that $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α.
- (Cardinal arithmetic) Silver's theorem asserts that if GCH fails at a singular strong limit cardinal κ of uncountable cofinality, then it fails for almost every $\mu < \kappa$.

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  These are both true for λ measurable, both false for (eg) $\lambda = \aleph_1$.

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- A *transversal* for a family of non-empty sets is a 1-1 choice function.

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- There's a substructure 0 which is minimal under inclusion.
- For any two substructures A, B there is a unique minimal substructure $A + B$ which contains $A \cup B$.
- The union of a continuous chain of substructures is a substructure.

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The crucial idea is to relativise the notion of freeness, that is to introduce a notion "B is free over A" where A is a substructure of B. The intention is that the free structures should be the ones which are free over 0. Typically the definition of B's being free over A will imply that any witness for A extends to a witness for B.

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- When A, B are non-empty families of countable sets and $A \subseteq B$, then B is free over A iff $B \setminus A$ has a transversal (say g) which takes values outside $\bigcup A$.

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Note: It's often true (and is true in our two running examples) that if C/A is free, then B/A for all B intermediate between A and C. But we don't need this.

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The rules are that $B_0 = 0$, A_i and (for $i > 0$) B_i have size κ, $B_0 \subseteq A_0 \subseteq B_1 \subseteq$ and B_{n+1}/B_n is free for all n. The first player to violate the rules loses, if the rules are followed forever then *II* wins.

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The rules are that C_i, D_i are structures of size κ and that $C_0 \subseteq D_0 \subseteq C_1 \ldots$. The first player to violate these rules loses, and if the rules are never violated then II wins iff $B' + D_\omega$ is free over $B + D_\omega$, where $D_\omega = \bigcup_{n<\omega} D_n = \bigcup_{n<\omega} C_n$.

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We also fix σ_i which is winning for II in $G_1(\lambda_i)$ when $i < \mu$ is not a limit ordinal, and $\tau_i(B, B')$ which is winning in $G_2(\lambda_i, \lambda_{i+1}, B, B')$ for all $i < \mu$ and all relevant B, B'.

Construction: we will build a matrix of substructures with ω rows and μ columns:

$$\begin{matrix} \vdots & \vdots & \vdots & \iddots \\ A_1^0 & A_1^1 & A_1^2 & \dots \\ B_1^0 & B_1^1 & B_1^2 & \dots \\ A_0^0 & A_0^1 & A_0^2 & \dots \\ B_0^0 = 0 & B_0^1 = 0 & B_0^2 = 0 & \dots \end{matrix}$$

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where:

- $|A^i_n| = |B^i_{n+1}| = \lambda_i$ for all $i < \mu$, $n < \omega$. $\bigcup_{i<\mu} A^i_0 = \mathcal{M}$.
- Column i is increasing for all $i < \mu$. As a consequence, $\bigcup_{n<\omega} A^i_n = \bigcup_{n<\omega} B^i_n$. We will denote this structure by B^i_ω.

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- The "n^{th} A-row" $(A_n^i)_{i<\mu}$ is increasing and continuous for all $n < \omega$. As a consequence, the sequence $(B_\omega^i)_{i<n}$ is increasing and continuous with union $\mathcal{M}$.
- For non-limit i, column i is a run of the game $G_1(\lambda_i)$ where player II is playing the structures B_n^i according to the winning strategy σ_i. As a consequence, B_{n+1}^i / B_n^i is free for all $n < \omega$.

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- For every $i<\mu$ and every pair B_n^{i+1}, B_{n+1}^{i+1} in column $i+1$, there is a run of the game $G_2(\lambda_i, \lambda_{i+1}, B_n^{i+1}, B_{n+1}^{i+1})$

I	B_{n+1}^i		B_{n+2}^i		$\dots$
II		$D_0^{i,n}$		$D_1^{i,n}$	

  where II is playing according to the winning strategy $\tau_i(B_n^{i+1}, B_{n+1}^{i+1})$. As a consequence, $(B_{n+1}^{i+1}+B_\omega^i)/(B_n^{i+1}+B_\omega^i)$ is free.

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- For each $i < \mu$, $\bigcup_{n<\omega} D_{\omega \cdot i+n} = B^{i+1}_\omega + B^i_\omega = B^{i+1}_\omega = B^{i+1}_\omega + B^{i+2}_0 = D_{\omega \cdot (i+1)}$.

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- For each limit $j < \mu$, $\bigcup_{i<j, n<\omega} D_{\omega \cdot i+n} = B_\omega^j = D_{\omega \cdot j}$.

How do we do it? The main issue is that we need the A-rows to be continuously increasing, and we need every column (including limit columns) to be constructed according to strategies for the game G_2. It is here that λ being singular (in particular $\mu < \lambda_0$) will be crucial.

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- The first two rows are easy: $B_0^i = 0$ for all i, and $(A_0^i)_{i<\mu}$ is any increasing and continuous chain of substructures with $|A_0^i| = \lambda_i$ and $\bigcup_{i<\mu} A_0^i = \mathcal{M}$.

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- The "B-rows" with positive subscripts are also easy: for non-limit i we compute B^i_{n+1} from $A^i_0, \dots A^i_n$ and the strategy σ_i, for limit i let $B^i_{n+1} = A^i_n$.

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  $$\begin{array}{llll} \mathrm{I} & B^i_{m+1} & & \ldots B^i_{n+1} \\ & & & \ldots \\ \mathrm{II} & & D^{i,m}_0 & \ldots D^{i,m}_{n-m} \end{array}$$

  of the game $G_2(\lambda_i, \lambda_{i+1}, B^{i+1}_m, B^{i+1}_{m+1})$, where player II is playing according to the winning strategy $\tau_i(B^{i+1}_m, B^{i+1}_{m+1})$.

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of the game $G_2(\lambda_i, \lambda_{i+1}, B^{i+1}_m, B^{i+1}_{m+1})$, where player II is playing according to the winning strategy $\tau_i(B^{i+1}_m, B^{i+1}_{m+1})$. Define an auxiliary set C^i_{n+1} such that $B^i_{n+1} \subseteq C^i_{n+1}$, $|C^i_{n+1}| = \lambda_i$, and $D^{i,m}_{n-m} \subseteq C^i_{n+1}$ for all $m \leqslant n$.

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- Since $\mu < \lambda_0 \leqslant \lambda_i$, we see that $|A^i_{n+1}| = \lambda_i$. The other key points are that $B^i_{n+1} \subseteq C^i_{n+1} \subseteq A^i_{n+1}$, and that $(A^i_{n+1})_{i<\mu}$ is continuous and increasing with i.

This concludes the proof. As we see shortly, we will need λ to be a limit cardinal to see that we can win $G_1(\kappa)$ for $\kappa < \lambda$. We needed λ *singular* to do the “looking ahead to all subsequent columns” in the main construction.

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Key idea for the first game: If all (many) substructures of size κ^+ are free (that is free over 0) then II has a winning strategy for $G_1(\kappa)$. It's important that λ is a limit cardinal, since we want to win for unboundedly many $\kappa < \lambda$.

We appeal to the well-known Gale-Stewart theorem on the determinacy of open games. The game is closed for player II, so if II does not win then I wins with some strategy σ.
We fix some large regular θ and build a continuous increasing chain $(M_i)_{i<\kappa^+}$ of elementary substructures of H_θ such that:

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- $\mathcal{M}, \sigma \in M_0$.
- $\kappa + 1 \subseteq M_i$, $|M_i| = \kappa$ for all $i < \kappa^+$.
- $\langle M_i : i \leqslant j \rangle \in M_{j+1}$ for all $j < \kappa$.

Let $M_\infty = \bigcup_{i<\kappa^+} M_i$, so that by hypothesis $M_\infty \cap \mathcal{M}$ is free. By our added assumption about "free over", we can find (taking the first ω points of an appropriate club) a strictly increasing ω-sequence (B_n) of structures of size κ such that $B_0 = 0$, $B_n = M_{\alpha_n} \cap \mathcal{M}$ for increasing $\alpha_n < \kappa^+$, and B_{n+1}/B_n is free for all n.

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Now we build a run of the game where player II plays the B_n's and player I responds using σ.

I		$\sigma(B_0)$		$\sigma(B_0, B_1)$	
					$\dots$
II	B_0		B_1		

This is a legitimate run of the game because σ, $\mathcal{M}$ and the models M_{α_i} for $0 < i \leqslant n$ are all elements of $M_{\alpha_{n+1}}$. So $\sigma(B_0, \ldots B_n) \in M_{\alpha_{n+1}}$, and hence easily $\sigma(B_0, \ldots B_n) \subseteq B_{n+1}$.

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Remark: It was an overkill to assume that *all* substructures of $\mathcal{M}$ with size κ^+ are free.

Adding more axioms about freeness, Shelah proved a result about G_2 parallel to the one I proved for G_1. But it turns out that in many cases (including my two running examples) we don't need any assumption about the ambient structure $\mathcal{M}$ to prove that player II wins G_2.

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Abelian groups: Let X be a set of coset representatives for a basis of B'/B. After I plays C_n, II finds $X_n \subseteq X$ of size κ such that every element of $C_n \cap B'$ is congruent mod B to something in $\text{span}(X_n)$, and then lets $D_n = C_n + \text{span}(X_n)$. Now check $X \setminus \bigcup_n X_n$ gives coset representatives for a basis of $(B' + D_\omega)/(B + D_\omega)$.

Transversals: Let g be a transversal of $B' \setminus B$ which does not take any value in $\bigcup B$. After I plays C_n, II finds $D_n \supseteq C_n$ such that $|D_n| = \kappa$ and $g(x) \in \bigcup C_n \implies x \in D_n$, this is possible because g is 1-1. Now check that $g \restriction B' \setminus (B \cup D_\omega)$ does not take any value in $\bigcup D_\omega$.

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By the compactness theorem for first order logic, $PT(\lambda, \omega)$ holds for all λ.

Let μ and τ be regular cardinals with $\mu < \tau$, and let $S \subseteq \tau \cap \mathrm{cof}(\mu)$. A *ladder system on* S is a sequence $(x_\delta)_{\delta \in S}$ such that x_δ is cofinal in δ with order type μ.

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By Fodor, there is no transversal. An easy diagonalisation shows that every countable subset has a transversal.

For τ regular and uncountable, a *non-reflecting stationary subset (NRSS) of* τ is $S \subseteq \tau$ such that S is stationary, and $S \cap \gamma$ is non-stationary for all $\gamma \in \tau \cap \mathrm{cof}(> \omega)$.

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Proof by induction on γ. If $\gamma = \gamma_0 + 1$ then nothing to do unless $\gamma_0 \in S$, in which case apply IH for $\delta < \gamma_0$ and then use $\sup(x_\delta \cap x_{\gamma_0}) < \delta$ to ensure disjointness from x_{γ_0}.

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Remark: By same argument, if $S \subseteq \omega_1$ is non-stationary then a ladder system on S has disjoint tails, in particular it has a transversal.

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Proof: Let $(A_i)_{i<\kappa}$ be countable sets witnessing $NPT(\kappa, \aleph_1)$. Let $(x_\delta)_{\delta \in S}$ be a ladder system on S, enumerate x_δ as $x_\delta(i)$ for $i < \kappa$. Define $B_{\delta,i} = (\{\delta\} \times A_i) \cup \{x_\delta(i)\}$, and claim that $(B_{\delta,i})_{\delta \in S, i<\kappa}$ exemplify $NPT(\lambda, \aleph_1)$.

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For any regular κ, $\kappa^+ \cap \mathrm{cof}(\kappa)$ is NRSS of κ^+. Since we know $NPT(\aleph_1, \aleph_1)$, deduce $NPT(\aleph_n, \aleph_1)$ for $1 \leqslant n < \omega$.

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But Magidor showed that modulo large cardinals (ω supercompact cardinals) that consistently every stationary subset of $\aleph_{\omega+1}$ reflects.

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- There exist unbounded $A \subseteq \omega$, and a sequence $(f_\alpha)_{\alpha<\aleph_{\omega+1}}$ which is increasing and cofinal in $(\prod_{n\in A} \aleph_n, <^*)$. Adjusting A and f's we may assume that $0 \notin A$ and $f_\alpha(n) \in [\aleph_n, \aleph_{n+1})$.

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- Let $\alpha < \aleph_{\omega+1}$ be a limit ordinal. An *exact upper bound* for $(f_\beta)_{\beta<\alpha}$ is $g \in \prod_{n \in A} \aleph_n$ such that $\{h \in \prod_{n \in A} \aleph_n : h <^* g\} = \{h \in \prod_{n \in A} \aleph_n : \exists \beta < \alpha \; h <^* f_\beta\}$. If an eub exists it is unique mod finite.

- If $\mathrm{cf}(\alpha) > \omega$ and α is a point where an eub g exists with $\mathrm{cf}(g(n)) > \omega$ for all n, then $\mathrm{cf}(g(n)) = \mathrm{cf}(\alpha)$ for all large n. Such α are called *good*. α is good iff there are $I \subseteq \alpha$ unbounded and $m < \omega$ such that $(f_\beta(n))_{\beta \in I}$ is strict increasing for $m \leqslant n < \omega$.

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- There are stationarily many good points in each uncountable cofinality.

Let T be the stationary set of good points of cofinality $\aleph_1$. PCF theory gives structural information about $T \cap \gamma$ for $\gamma < \aleph_{\omega+1}$ with $\omega_1 < \mathrm{cf}(\gamma)$:

- If γ is good, then almost all points in $\gamma \cap \mathrm{cof}(\omega_1)$ are in T.
- If γ is ungood, then almost all points in $\gamma \cap \mathrm{cof}(\omega_1)$ are not in T.

Let T be the stationary set of good points of cofinality $\aleph_1$. PCF theory gives structural information about $T \cap \gamma$ for $\gamma < \aleph_{\omega+1}$ with $\omega_1 < \mathrm{cf}(\gamma)$:

- If γ is good, then almost all points in $\gamma \cap \mathrm{cof}(\omega_1)$ are in T.
- If γ is ungood, then almost all points in $\gamma \cap \mathrm{cof}(\omega_1)$ are not in T.

For the experts: If γ is good, fix I and n witnessing this: all α of cofinality ω_1 such that I is unbounded in α are good. If γ is ungood, it is in the Bad or Ugly cases of Shelah's trichotomy: in either case the witnessing objects witness ungoodness almost everywhere below.

Viewed as sets of ordered pairs, the f_α's form an almost disjoint family of countable subsets of $A \times \aleph_\omega$. To emphasise that we are think of them as sets, we write $A_\alpha = \{(m, f_\alpha(m)) : \alpha \in A\}$. Ordering A_α by first entries, we have a notion of "tail of A_α".

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Idea of proof: Construct a witness to $NPT(\aleph_{\omega+1}, \aleph_2)$ and then "step down" to get a witness to $NPT(\aleph_{\omega+1}, \aleph_1)$.

Key claim: for all $\gamma < \aleph_{\omega+1}$ there exist (B_α, D_α) for $\alpha \in T \cap \gamma$ such that:

- B_α is a tail of A_α.
- D_α is club in α with $\mathrm{ot}(D_\alpha) = \omega_1$.
- The sets $B_\alpha \times D_\alpha$ are pairwise disjoint.

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Assuming key claim, we fix E_α club in α for each $\alpha \in T$ and claim that $\{A_\alpha \times E_\alpha : \alpha \in T\}$ exemplify $NPT(\kappa^+, \aleph_2)$. There is no transversal of the whole system (freeze 1st coordinate on a stationary set, then apply Fodor on 2nd coordinate). For $\gamma < \aleph_{\omega+1}$ apply the key claim and see that $B_\alpha \times (D_\alpha \cap E_\alpha) \subseteq A_\alpha \times E_\alpha$, these subsets are nonempty and pairwise disjoint.

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Easy case 1: $\gamma = \gamma_0 + 1$. Apply IH to γ_0, and then if $\gamma_0 \in S$ choose D_{γ_0} and replace D_α's below by tails disjoint from D_{γ_0}.

Easy case 2: There exist $\gamma_i \notin T$ increasing continuous and cofinal in γ. Use γ_i's to cut γ into blocks, apply IH in each block, then replace D_α for $\alpha \in [\gamma_i, \gamma_{i+1})$ by tail above γ_i.

Hard case: None of the above. By assumption γ is good and $\mathrm{cf}(\gamma) > \omega_1$. Fix $I \subseteq \gamma$ cofinal and m such that $(f_\alpha(n))_{\alpha \in I}$ is increasing for $n \geqslant m$.

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For $\alpha \in S \cap \lim(C)$ choose $D_\alpha = C \cap \alpha$, so that $D_\alpha \cap D_\beta = \emptyset$ for $\beta < \alpha$ unless also $\beta \in S \cap \lim(C)$.

Key point: Fix $\alpha \in S \cap \lim(C)$. For every $\beta \in I \cap \alpha$, there is $n(\beta) \geqslant m$ such that $f_\beta(n) < f_\alpha(n)$ for $n \geqslant n(\beta)$. As $\mathrm{cf}(\alpha) = \omega_1$, there is $J \subseteq I \cap \alpha$ unbounded and n^* such that $n(\beta) = n^*$ for $\beta \in J$. But then (by choice of I and m) $f_\beta(n) < f_\alpha(n)$ for *all* $\beta \in I \cap \alpha$ and $n \geqslant n^*$.

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Let $\eta(\alpha)$ be the least point of I above α. Then we can choose $m(\alpha) \geqslant m$ such that for $n \geqslant m(\alpha)$:

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It follows that $B_\alpha \cap B_\beta = \emptyset$.