Extremely amenable automorphism groups

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- The relation between the (EA) of $\mathrm{Aut}(\mathcal{M})$ and the (approximate) Ramsey properties of $\mathrm{Age}(\mathcal{M})$ (the KPT-correspondence).

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- The "metric" theory for the case of Banach spaces.
- The Gurarij space and the $L_p[0, 1]$-spaces.

Part I: Basics

1. Topological Dynamics
   - Extreme Amenability, Universal Minimal Flows
   - UMF vs EA; how to prove EA
2. (Metric) Fraïssé Theory
   - First order structures
   - KPT correspondence; Structural Ramsey Properties
   - Structural Ramsey Theorems
   - Metric structures

Part II: An example of metric structures: Banach spaces

Part III: Three Examples

6. Gurarij space
   - $\{\ell_\infty^n\}_n$ have have the (ARP)
   - The ARP of Finite dimensional Normed spaces
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7. L_p-spaces
   - L_p (sometimes) is a Fraïssé space
   - $\{\ell_p^n\}$ have the (ARP)

Extreme Amenability

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Definition

A topological group G is called extremely amenable (EA) when every continuous action (flow) $G \curvearrowright K$ on a compact K has a fixed point; that is, there is $p \in K$ such that $g \cdot p = p$ for all $g \in G$.

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EA groups are amenable (G is amenable iff every affine flow $G \curvearrowright K$ on a compact convex space K has a fixed point).

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is ... $\phi(x)$.

We consider the commutative C^*-algebra of right uniformly continuous and bounded $f : G \to \mathbb{C}$, and represent it as $C(S(G))$ (Gelfand); any minimal flow of $S(G)$ is G-isomorphic to $\mathcal{M}(G)$.

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Compute universal minimal flows.

Examples of EA groups

1. The unitary group $\mathbb{U}$ of linear isometries of the separable infinite dimensional Hilbert space $\mathbb{H}$, endowed with its strong operator topology SOT (i.e. the pointwise convergence topology) (Gromov-Milman);

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3. The group of isometries of the Urysohn space with its pw. conv. top. (Pestov);

4. The group of linear isometries of the Lebesgue spaces $L_p[0, 1]$, $1 \leq p \neq 2 < \infty$, with the SOT (Giordano-Pestov);

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6. The Automorphism group of the ordered universal $\mathbb{F}$-vector space $\mathbb{F}^{<\infty}$, $\mathbb{F}$ finite field, is extremely amenable (K-P-T);

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7. The group of linear isometries of the Gurarij space $\mathbb{G}$ (Bartosova-LA-Lupini-Mbombo).

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4. $M(\mathrm{Aut}(\mathbb{P})) = \mathbb{P}$, where $\mathbb{P}$ is the Poulsen simplex, the unique compact metrizable Choquet simplex whose extreme points are dense (B-LA-L-M).

UMF and EA

Proposition (Ben Yaacov-Melleray-Tsankov)

Suppose that G is a polish group (i.e. separable and complete metrizable topological group). If the umf $M(G)$ is metrizable, then there is an EA subgroup H of G such that $M(G)$ is the completion of G/H.

Proving that a group is EA

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While the first seems a restricted approach, the second is general, as proved by Melleray.

$\mathrm{Aut}(X)$ is extremely amenable

X	**Method**
$\mathbb{H}$	Lévy
$\mathbb{Q}$	KPT
$\mathbb{U}$	Lévy and KPT
$L_p[0,1]$	Lévy and KPT
$\mathbb{B}$	KPT
$\mathbb{F}^{<\infty}$	KPT
$\mathbb{G}$	KPT

Table: Methods to prove extreme amenability

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Definition (Ultrahomogeneity)

A first order structure $\mathcal{M}$ is called ultrahomogeneous when for every finitely generated substructure $\mathcal{N}$ of $\mathcal{M}$ and every embedding $\phi : \mathcal{N} \to \mathcal{M}$ there is an automorphism $g \in \mathrm{Aut}(\mathcal{M})$ such that $g \upharpoonright N = \phi$.

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Fraïssé theory tells that countable ultrahomogeneous structures are the Fraïssé limits of Fraïssé classes (hereditary property, joint embedding property, and amalgamation property).

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Proposition (Representation Theorem I)

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Permutations of $\mathbb{N}$ with the topology of point-wise convergence.

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Proof.

For suppose that G is a closed subgroup of $\mathcal{S}_\infty$; For each $k \in \mathbb{N}$, consider the canonical action $G \curvearrowright \mathbb{N}^k$, $g \cdot (a_j)_{j<k} := (g(a_j))_{j<k}$, and let $\{O_j^{(k)}\}_{j \in I_k}$ be the enumeration of the corresponding orbits. Let $\mathcal{L}$ be the relational language, $\{R_j^{(k)} \,:\, k \in \mathbb{N}, j \in I_k\}$, each $R_j^{(k)}$ being a k-ari relational symbol. Now $\mathbb{N}$ is an $\mathcal{R}$-structure $\mathcal{M}$ naturally,

$$(R_j^{(k)})^{\mathcal{M}} := O_j^{(k)}.$$

It is easy to see that $\mathcal{M}$ is ultrahomogeneous, and that $G \subseteq \mathrm{Aut}(\mathcal{M})$ is dense in G, so, equal to G.

□

Kechris-Pestov-Todorcevic correspondence

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Definition (Structural Ramsey Property)

Let $\mathcal{F}$ be a class of finitely generated first order structures of the same sort. The class $\mathcal{F}$ has the **Structural** Ramsey Property (RP) if for every $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ and every $r \in \mathbb{N}$ there is $\mathbf{C} \in \mathcal{F}$ such that for every coloring $c : \mathrm{emb}(\mathbf{A}, \mathbf{C}) \to r$ there is $\varrho \in \mathrm{emb}(\mathbf{B}, \mathbf{C})$ such that $\varrho \circ \mathrm{emb}(\mathbf{A}, \mathbf{B})$ is c-monochromatic.

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Let M be a countable ultrahomogeneous structure. TFAE:

1. $\mathrm{Aut}(M)$ *is extremely amenable;*
2. $\mathrm{Age}(M)$ *has the Ramsey property (RP).*

The Classical Ramsey Theorem

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Proposition (F. P. Ramsey)

For every $k, m, r \in \mathbb{N}$ there is $n \geq k$ such that every r-coloring

$$c : [n]^k \to r$$

has a monochromatic set of the form $[A]^k$ for some $A \subseteq n$ of cardinality m.

This is equivalent to the following: Let $\mathrm{emb}(k, n)$ be the collection of all injections $f : k \to n$ (so, no structure).

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1. *The class of finite linear orderings has the Ramsey property, and*
2. $\mathrm{Aut}(\mathbb{Q},<)$ *is extremely amenable.*

The Dual Ramsey Theorem (DRT)

Let $\mathcal{E}_n^d$ be the set of all partitions of n into d-many pieces. Given a partition $\mathcal{Q} \in \mathcal{E}_n^m$, and $d \leq m$, let $\langle \mathcal{Q} \rangle^d$ be set of all partitions $\mathcal{P} \in \mathcal{E}_n^d$ *coarser* than $\mathcal{Q}$.

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Theorem (Dual Ramsey by Graham and Rothschild)

For every d, m and r there exists n such that for every coloring $c : \mathcal{E}_n^d \to r$ there exists $\mathcal{Q} \in \mathcal{E}_n^m$ such that $c \restriction \langle \mathcal{Q} \rangle^d$ is constant.

By a simple dual argument, this is equivalent to the following. Given $k, n \in \mathbb{N}$, we consider $\mathcal{P}(k)$ and $\mathcal{P}(n)$ as boolean algebras, and then let $\mathrm{emb}(k, n)$ be the collection of all ordered boolean embeddings $f : \mathcal{P}(k) \to \mathcal{P}(n)$, i.e., such that $\min f(\{i\} < \min f(\{j\})$ for every $i < j < k$. The dual Ramsey theorem can be restated as follows.

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Theorem (DR, Boolean version)

For every k, m and r in $\mathbb{N}$ there is some $n \in \mathbb{N}$ such that every r-coloring $c : \mathrm{emb}(\mathcal{P}(k), \mathcal{P}(n)) \to r$ has a monochromatic set of the form $\varrho \circ \mathrm{emb}(\mathcal{P}(k), \mathcal{P}(m))$ for some $\varrho \in \mathrm{emb}(\mathcal{P}(m), \mathcal{P}(n))$; consequently,

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1. *The class of finite, canonically ordered, boolean algebras has the Ramsey property, and*
2. *The automorphism group of the canonically ordered countable atomless boolean algebra is extremely amenable.*

Metric structures

The rest of the examples are also groups of algebraic automorphisms that are in addition isometries. First order structures are the discrete version of metric structures $\mathcal{M} = (M, (F^{\mathcal{M}})_{F\in\mathcal{F}}, (R^{\mathcal{M}})_{R\in\mathcal{F}})$: Roughly speaking:

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1. metric spaces,
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4. operator spaces, etc.

Approximate Ultrahomogeneity

Definition (Approximate Ultrahomogeneity)

A metric structure $\mathcal{M}$ is called approximate ultrahomogeneous when for every finitely generated substructure $\mathcal{N}$ of $\mathcal{M}$ and every embedding $\phi : \mathcal{N} \to \mathcal{M}$ there is an automorphism $g \in \mathrm{Aut}(\mathcal{M})$ such that $\widehat{d}(g \restriction N, \phi) < \varepsilon$.

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Metric Fraïssé theory tells that countable ultrahomogeneous structures are the Fraïssé limits of Fraïssé classes (hereditary property, joint embedding property, and near amalgamation property).

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Proposition (Representation Theorem II; Melleray)

Every polish group G is the automorphism group of an approximate ultrahomogeneous metric structure.

Metric KPT correspondence

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Ramsey theorem, Dual Ramsey Theorem, Graham-Leeb-Rothschild, Nešetřil...

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$\mathbb{Q}$ as order, $\mathbb{B}_{<\infty}$, $\mathbb{F}^{<\infty}$, $\mathbb{U}$

An example of metric structures: Banach spaces

Outline

Fraïssé Banach spaces

ARP for finite dimensional normed spaces

Given two Banach spaces X and Y, and $\delta \geq 0$, let $\mathrm{Emb}_\delta(X, Y)$ be the collection of all linear 1-1 bounded functions $T : X \to Y$ such that $\|T\|, \|T^{-1}\| \leq 1 + \delta$.

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Definition

A collection $\mathcal{F}$ of finite dimensional normed spaces has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathcal{F}$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that every continuous coloring c of $\mathrm{Emb}(F, H)$ ε-stabilizes in $\varrho \circ \mathrm{Emb}(F, G)$ for some $\varrho \in \mathrm{Emb}(G, H)$, that is,

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This is a particular instance of a more general definition for metric structures.

Comparing different Ramsey Propeties

****** **falta** ****

1. arp
2. compact arp
3. discrete arp
4. Property for $\mathrm{Emb}(X, E)$.

Three examples

Outline

6 Gurarij space
   - $\{\ell_\infty^n\}_n$ have have the (ARP)
   - The ARP of Finite dimensional Normed spaces
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7 L_p-spaces
   - L_p (sometimes) is a Fraïssé space
   - $\{\ell_p^n\}$ have the (ARP)

First application of metric KPT correspondence

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There are also noncommutative analogues. ****** **falta** **** mention M-spaces.

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