LARGE CARDINALS IN GENERAL TOPOLOGY III

Miroslav HUŠEK

Winter School in Abstract Analysis
Hejnice, January 26–February 2, 2019
Locally presentable categories are precisely those categories which can be axiomatized by limit sentences in an infinitary first-order logic. They include varieties and quasivarieties of algebras.
**Definition**

Let $X = \lim\left\{ X_i, \pi_{i,j}\right\}_I$. A map $f : X \to A$ is said to depend on a (down-directed) set $J \subset I$ if there exists a map $g : \lim\left\{ \text{pr}_i(X_i), \pi_{i,j}\right\}_J \to A$ in $\mathcal{C}$ such that $f = gp_J$.

**Task**

Let $\mathcal{C}$ be generated by a space $A$. We are looking for a cardinal $\lambda$ such that every map from $\lim\left\{ X_i, \pi_{i,j}\right\}_I$ into $A$ depends on some $J \subset I$ with $|J| < \lambda$.

Such a least cardinal $\lambda$ is denoted by $\text{coord}(\mathcal{C}, A)$.

**Special task**

When discrete ordered index sets $I$ are used only, notations $\text{coord}_d(\mathcal{C}, A)$ are used.
**Definition**

Let $X = \lim\{X_i, \pi_{i,j}\}_I$. A map $f : X \to A$ is said to depend on a (down-directed) set $J \subset I$ if there exists a map $g : \lim\{\text{pr}_i(X_i), \pi_{i,j}\}_J \to A$ in $C$ such that $f = gp_J$.

**Task**

Let $C$ be generated by a space $A$. We are looking for a cardinal $\lambda$ such that every map from $\lim\{X_i, \pi_{i,j}\}_I$ into $A$ depends on some $J \subset I$ with $|J| < \lambda$.

Such a least cardinal $\lambda$ is denoted by $\text{coord}(C, A)$.

**Special task**

When discrete ordered index sets $I$ are used only, notations $\text{coord}_d(C, A)$ are used.
**Definition**

Let $X = \lim_{I} \{X_i, \pi_{i,j}\}$. A map $f : X \to A$ is said to depend on a (down-directed) set $J \subset I$ if there exists a map $g : \lim_{J} \{\text{pr}_i(X_i), \pi_{i,j}\} \to A$ in $C$ such that $f = gp_J$.

**Task**

Let $C$ be generated by a space $A$. We are looking for a cardinal $\lambda$ such that every map from $\lim_{I} \{X_i, \pi_{i,j}\}$ into $A$ depends on some $J \subset I$ with $|J| < \lambda$.

Such a least cardinal $\lambda$ is denoted by coord($C, A$).

**Special task**

When discrete ordered index sets $I$ are used only, notations coord$_d(C, A)$ are used.
Theorem

Let \( X = \lim_{\leftarrow} \{ X_i, \pi_{i,j} \} \) and \( f : X \to A \), \( J \) be a down-directed subset of \( I \).
Then \( f \) depends on \( J \) iff \( f \) depends on \( J \) regarding \( X \) as a subspace of \( \prod_I X_i \) and the factorized map extends in \( C \) onto \( \text{pr}_J(X) \).

Theorem

coordinates \( d(C, A) \leq \lambda \) iff for any family \( \{ X_i \}_I \subset C \), any closed subset \( X \subset \prod_I X_i \) and any map \( f : X \to A \), the map \( f \) depends on some \( J \subset I \) with \( |J| < \lambda \) and the factorized map can be extended to a continuous map \( \text{pr}_J(X) \to A \).

Theorem

coordinates \( d(C, A) \leq \lambda \) iff for any family \( \{ X_i \}_I \subset C \) and any map \( f : \prod_I X_i \to A \), the map \( f \) factorizes in \( C \) via a subproduct \( \prod_J X_i \) for some \( J \subset I \) with \( |J| < \lambda \).
Theorem

Let $X = \lim \{X_i, \pi_{i,j}\}_I$ and $f : X \to A$, $J$ be a down-directed subset of $I$. Then $f$ depends on $J$ iff $f$ depends on $J$ regarding $X$ as a subspace of $\prod_I X_i$ and the factorized map extends in $\mathcal{C}$ onto $\text{pr}_J(X)$.

Theorem

$\text{coord} (\mathcal{C}, A) \leq \lambda$ iff for any family $\{X_i\}_I \subset \mathcal{C}$, any closed subset $X \subset \prod_I X_i$ and any map $f : X \to A$, the map $f$ depends on some $J \subset I$ with $|J| < \lambda$ and the factorized map can be extended to a continuous map $\text{pr}_J(X) \to A$.

Theorem

$\text{coord}_d (\mathcal{C}, A) \leq \lambda$ iff for any family $\{X_i\}_I \subset \mathcal{C}$ and any map $f : \prod_I X_i \to A$, the map $f$ factorizes in $\mathcal{C}$ via a subproduct $\prod_J X_i$ for some $J \subset I$ with $|J| < \lambda$. 
Theorem

Let $X = \lim \left\{ X_i, \pi_{i,j} \right\}_I$ and $f : X \to A$, $J$ be a down-directed subset of $I$. Then $f$ depends on $J$ iff $f$ depends on $J$ regarding $X$ as a subspace of $\prod_I X_i$ and the factorized map extends in $C$ onto $\text{pr}_J (X)$.

Theorem

$\text{coord}(C, A) \leq \lambda$ iff for any family $\{X_i\}_I \subset C$, any closed subset $X \subset \prod_I X_i$ and any map $f : X \to A$, the map $f$ depends on some $J \subset I$ with $|J| < \lambda$ and the factorized map can be extended to a continuous map $\text{pr}_J (X) \to A$.

Theorem

$\text{coord}_d(C, A) \leq \lambda$ iff for any family $\{X_i\}_I \subset C$ and any map $f : \prod_I X_i \to A$, the map $f$ factorizes in $C$ via a subproduct $\prod_J X_i$ for some $J \subset I$ with $|J| < \lambda$. 
Let $m_C$ be the least cardinal $\kappa$ such that a discrete space of cardinality $\kappa$ does not belong to $C$. In fact, 
$m_C = \min\{\kappa; \text{there exists } X \in C \text{ that is not pseudo-}\kappa\text{-compact}\}$

The cardinal $m_C$ is measurable or equals to $\infty$.
Always $\text{coord}(C, A) \geq \text{coord}_d(C, A) \geq m_C$
If $C$ is simple and $m_C < \infty$ then $\text{coord}_d(C) < \infty$.

$m_C > \omega$

If $m_0 < m_C < \infty$ then $\text{coord}_d(C, A) = m_C$ provided $\bar{\psi}(\Delta_A) < m_C$.
Thus $\text{coord}_d(\text{RealComp}) = m_1$. 
Let $m_C$ be the least cardinal $\kappa$ such that a discrete space of cardinality $\kappa$ does not belong to $\mathcal{C}$. In fact,
$$m_C = \min\{\kappa; \text{there exists } X \in \mathcal{C} \text{ that is not pseudo-}\kappa\text{-compact}\}$$

The cardinal $m_C$ is measurable or equals to $\infty$.

Always $\text{coord}(\mathcal{C}, A) \geq \text{coord}_d(\mathcal{C}, A) \geq m_C$

If $\mathcal{C}$ is simple and $m_C < \infty$ then $\text{coord}_d(\mathcal{C}) < \infty$.

$m_C > \omega$

If $m_0 < m_C < \infty$ then $\text{coord}_d(\mathcal{C}, A) = m_C$ provided $\overline{\psi}(\Delta_A) < m_C$.
Thus $\text{coord}_d(\text{RealComp}) = m_1$. 


Let $m_{c}$ be the least cardinal $\kappa$ such that a discrete space of cardinality $\kappa$ does not belong to $C$. In fact,

$m_{c} = \min\{\kappa; \text{there exists } X \in C \text{ that is not pseudo-}\kappa\text{-compact}\}$

The cardinal $m_{c}$ is measurable or equals to $\infty$.
Always $\text{coord}(C, A) \geq \text{coord}_{d}(C, A) \geq m_{c}$
If $C$ is simple and $m_{c} < \infty$ then $\text{coord}_{d}(C) < \infty$.

$m_{c} > \omega$

If $m_{0} < m_{c} < \infty$ then $\text{coord}_{d}(C, A) = m_{c}$ provided $\varPsi(\Delta_{A}) < m_{c}$.
Thus $\text{coord}_{d}(\text{RealComp}) = m_{1}$.
Let $m_C$ be the least cardinal $\kappa$ such that a discrete space of cardinality $\kappa$ does not belong to $C$. In fact,

$$m_C = \min\{ \kappa; \text{there exists } X \in C \text{ that is not pseudo-}\kappa\text{-compact} \}$$

The cardinal $m_C$ is measurable or equals to $\infty$.

Always $\text{coord}(C, A) \geq \text{coord}_d(C, A) \geq m_C$

If $C$ is simple and $m_C < \infty$ then $\text{coord}_d(C) < \infty$.

$m_C > \omega$

If $m_0 < m_C < \infty$ then $\text{coord}_d(C, A) = m_C$ provided $\bar{\psi}(\Delta_A) < m_C$.

Thus $\text{coord}_d(\text{RealComp}) = m_1$. 
The cardinal $m_C$ is measurable or equals to $\infty$.
Always $\text{coord}(C, A) \geq \text{coord}_d(C, A) \geq m_C$
If $C$ is simple and $m_C < \infty$ then $\text{coord}_d(C) < \infty$.

**Theorem (N.Noble, M.Ulmer 1972)**

*Let $\kappa$ be a regular cardinal. If $\prod_I X_i$ is pseudo-$\kappa$-compact then every continuous $f : \prod_I X_i \to A$ depends on less than $\kappa$ coordinates for any $A$ with $\overline{\psi}(\Delta_A) < \kappa$.

If all $X_i$ are completely regular and $\prod_I X_i$ is not pseudo-$\kappa$-compact, where $\text{cof}(\kappa) > \omega$, then there exists a continuous real-valued function on the product not depending on less than $\kappa$ coordinates.*

$m_C > \omega$

If $m_0 < m_C < \infty$ then $\text{coord}_d(C, A) = m_C$ provided $\overline{\psi}(\Delta_A) < m_C$.
Thus $\text{coord}_d(\text{RealComp}) = m_1$. 
The cardinal $m_C$ is measurable or equals to $\infty$.
Always $\operatorname{coord}(C, A) \geq \operatorname{coord}_d(C, A) \geq m_C$
If $C$ is simple and $m_C < \infty$ then $\operatorname{coord}_d(C) < \infty$.

$m_C > \omega$
If $m_0 < m_C < \infty$ then $\operatorname{coord}_d(C, A) = m_C$ provided $\overline{\psi}(\Delta_A) < m_C$.
Thus $\operatorname{coord}_d(\text{RealComp}) = m_1$. 
Definition
A regular cardinal $\kappa$ is said to be $\lambda$-strongly compact for $\omega < \lambda \leq \kappa$ if every $\kappa$-complete filter on any set has an extension to $\lambda$-complete ultrafilter.

Every $\lambda$-strongly compact cardinal is $\mu$-strongly compact for any infinite $\mu \leq m(\lambda)$.

Every $\lambda$-strongly compact cardinal is measurable and the first uncountable measurable cardinal $m_1$ may be $\omega_1$-strongly compact.

If $\kappa$ is a $\lambda$-strongly compact cardinal then $\nu_\lambda(\mu) \cap U_\mu \neq \emptyset$ for every $\mu$ with $\text{cof}(\mu) \geq \kappa$.

In particular, if $m_1$ is $\omega_1$-strongly compact then $\nu(\mu) \cap U_\mu \neq \emptyset$ for every $\mu$ with $\text{cof}(\mu) \geq m_1$ (clearly, $\nu(\mu) \cap U_\mu = \emptyset$ if $\text{cof}(\mu) < m_1$).
Definition

A regular cardinal $\kappa$ is said to be $\lambda$-strongly compact for $\omega < \lambda \leq \kappa$ if every $\kappa$-complete filter on any set has an extension to $\lambda$-complete ultrafilter.

Every $\lambda$-strongly compact cardinal is $\mu$-strongly compact for any infinite $\mu \leq \text{m}(\lambda)$.

Every $\lambda$-strongly compact cardinal is measurable and the first uncountable measurable cardinal $\text{m}_1$ may be $\omega_1$-strongly compact.

If $\kappa$ is a $\lambda$-strongly compact cardinal then $\nu_{\lambda}(\mu) \cap U_\mu \neq \emptyset$ for every $\mu$ with $\text{cof}(\mu) \geq \kappa$.

In particular, if $\text{m}_1$ is $\omega_1$-strongly compact then $\nu(\mu) \cap U_\mu \neq \emptyset$ for every $\mu$ with $\text{cof}(\mu) \geq \text{m}_1$ (clearly, $\nu(\mu) \cap U_\mu = \emptyset$ if $\text{cof}(\mu) < \text{m}_1$).
Definition
A regular cardinal $\kappa$ is said to be $\lambda$-strongly compact for $\omega < \lambda \leq \kappa$ if every $\kappa$-complete filter on any set has an extension to $\lambda$-complete ultrafilter.

Every $\lambda$-strongly compact cardinal is $\mu$-strongly compact for any infinite $\mu \leq m(\lambda)$.

Every $\lambda$-strongly compact cardinal is measurable and the first uncountable measurable cardinal $m_1$ may be $\omega_1$-strongly compact.

If $\kappa$ is a $\lambda$-strongly compact cardinal then $\nu_\lambda(\mu) \cap U_\mu \neq \emptyset$ for every $\mu$ with $\text{cof}(\mu) \geq \kappa$.

In particular, if $m_1$ is $\omega_1$-strongly compact then $\nu(\mu) \cap U_\mu \neq \emptyset$ for every $\mu$ with $\text{cof}(\mu) \geq m_1$ (clearly, $\nu(\mu) \cap U_\mu = \emptyset$ if $\text{cof}(\mu) < m_1$).
Theorem

Let $\mu$-strongly compact cardinals exist and $\kappa$ be the smallest one. Let $A$ be such a space that every $\mu$-complete ultrafilter on any set $D$ converges in a reflection $r_A(D)$ of $D$ in $A$-compact spaces. Then $\text{coord}(\mathcal{C}(A), A) \leq \max(\kappa, \chi(A)^+)$. 

Theorem (Realcompact spaces)

If $\omega_1$-strongly compact cardinals exist and $\lambda$ is the smallest one then for $C$ composed of realcompact spaces and containing $\mathbb{N}$ one has $m_1 \leq \text{coord}(C) \leq \lambda$.

Corollary

If $m_1$ is $\omega_1$-strongly compact then for $C$ composed of realcompact spaces and containing $\mathbb{N}$ one has $\text{coord}(C) = m_1$. 
**Theorem**

Let \( \mu \)-strongly compact cardinals exist and \( \kappa \) be the smallest one. Let \( A \) be such a space that every \( \mu \)-complete ultrafilter on any set \( D \) converges in a reflection \( r_A(D) \) of \( D \) in \( A \)-compact spaces. Then \( \text{coord}(\mathcal{C}(A), A) \leq \max(\kappa, \chi(A)^+) \).

**Theorem (Realcompact spaces)**

If \( \omega_1 \)-strongly compact cardinals exists and \( \lambda \) is the smallest one then for \( C \) composed of realcompact spaces and containing \( \mathbb{N} \) one has \( m_1 \leq \text{coord}(\mathcal{C}) \leq \lambda \).

**Corollary**

If \( m_1 \) is \( \omega_1 \)-strongly compact then for \( C \) composed of realcompact spaces and containing \( \mathbb{N} \) one has \( \text{coord}(\mathcal{C}) = m_1 \).
**Theorem**

Let $\mu$-strongly compact cardinals exist and $\kappa$ be the smallest one. Let $A$ be such a space that every $\mu$-complete ultrafilter on any set $D$ converges in a reflection $r_A(D)$ of $D$ in $A$-compact spaces. Then $\text{coord}(\mathcal{C}(A), A) \leq \max(\kappa, \chi(A)^+)$.  

**Theorem (Realcompact spaces)**

If $\omega_1$-strongly compact cardinals exists and $\lambda$ is the smallest one then for $\mathcal{C}$ composed of realcompact spaces and containing $\mathbb{N}$ one has $m_1 \leq \text{coord}(\mathcal{C}) \leq \lambda$.  

**Corollary**

If $m_1$ is $\omega_1$-strongly compact then for $\mathcal{C}$ composed of realcompact spaces and containing $\mathbb{N}$ one has $\text{coord}(\mathcal{C}) = m_1$. 
Proof

Theorem

Let $\mu$-strongly compact cardinals exist and $\kappa$ be the smallest one. Let $A$ be such a space that every $\mu$-complete ultrafilter on any set $D$ converges in a reflection $r_A(D)$ of $D$ in $A$-compact spaces, and let $\chi(A) < \kappa$. Then $\text{coord}(C(A), A) \leq \kappa$. 
Proof

Theorem

Let $\mu$-strongly compact cardinals exist and $\kappa$ be the smallest one. Let $A$ be such a space that every $\mu$-complete ultrafilter on any set $D$ converges in a reflection $r_A(D)$ of $D$ in $A$-compact spaces, and let $\chi(A) < \kappa$. Then $\text{coord}(C(A), A) \leq \kappa$.

1. Let $f : X \to A$, $X$ is a closed subspace of $A^I$, does not depend on less than $\kappa$ coordinates. Then for every $J \in [I]^{<\kappa}$ the following sets are nonempty:

$$C_J = \{ x \in X; \text{ there is } y \in X \text{ such that } \text{pr}_J(x) = \text{pr}_J(y), f(x) \neq f(y) \}.$$ 

The sets $C_J$ form a base of a $\kappa$-complete filter $\mathcal{F}$ in the set $X$. By our assumption, $\mathcal{F}$ can be extended to a $\mu$-complete ultrafilter on the set $X$ and it has an accumulation point $\xi$ in the space $X$. 
Proof

Theorem

Let $\mu$-strongly compact cardinals exist and $\kappa$ be the smallest one. Let $A$ be such a space that every $\mu$-complete ultrafilter on any set $D$ converges in a reflection $r_A(D)$ of $D$ in $A$-compact spaces, and let $\chi(A) < \kappa$. Then $\text{coord}(\mathcal{C}(A), A) \leq \kappa$.

2. Now $f$ depends on some $J_0 \in [I]^{<\kappa}$ and $f = f_J \text{pr}_J$ for any $J \in \mathcal{J} = \{ J \in [I]^{<\kappa}, J \supset J_0 \}$. Assume no such $f_J$ is continuous and denote

$$C_J = \{ x \in X; f_J \text{ is not continuous at } \text{pr}_J(x) \}.$$ 

Again, the sets $C_J$ form a base of a $\kappa$-complete filter $\mathcal{F}$ in the set $X$ extendible to a $\mu$-complete ultrafilter on the set $X$ and, thus, having an accumulation point in $X$. 
Proof

**Theorem**

Let \( \mu \)-strongly compact cardinals exist and \( \kappa \) be the smallest one. Let \( A \) be such a space that every \( \mu \)-complete ultrafilter on any set \( D \) converges in a reflection \( r_A(D) \) of \( D \) in \( A \)-compact spaces, and let \( \chi(A) < \kappa \). Then \( \text{coord}(\mathcal{C}(A), A) \leq \kappa \).

3. Now \( f \) depends continuously on some \( J_0 \in [\mathcal{I}]^{<\kappa} \). Denote again \( \mathcal{J} = \{ J \in [\mathcal{I}]^{<\kappa}, J \supset J_0 \} \). Assume no such \( f_J \) can be continuously extended to \( \text{pr}_J(X) \). Then we can define

\[
C_J = \{ x \in \text{pr}_J^{-1}(\text{pr}_J(X)); f_J \text{ does not extend continuously to } \text{pr}_J(x) \}.
\]

Again, the sets \( C_J \) form a base of a \( \kappa \)-complete filter \( \mathcal{F} \) in \( A^\mathcal{I} \) extendible to a \( \mu \)-complete ultrafilter on \( A^\mathcal{I} \) and, thus, having an accumulation point in \( X \).
Herrlich’s $\kappa$-compact spaces

Let $\kappa$ be an infinite cardinal. The class of $\kappa$-compact spaces is denoted by $\mathcal{H}(\kappa)$.

We repeat that a Tikhonov space is $\kappa$-compact if every maximal zero-filter, that is $\kappa$-complete, is fixed; $[0, 1]^{\kappa} \setminus \{1\}$-compact spaces are exactly $\kappa^+$-compact spaces.

If $\kappa$ is not measurable then $m_{\mathcal{H}(\kappa)} = m(\kappa)$

Theorem

$m(\kappa) \leq \text{coord}(\mathcal{H}(\kappa)) \leq \inf \{\kappa; \kappa \text{ is } \kappa\text{-strongly compact}\}$.
Herrlich’s $\kappa$-compact spaces

Let $\kappa$ be an infinite cardinal. The class of $\kappa$-compact spaces is denoted by $\mathcal{H}(\kappa)$.

We repeat that a Tikhonov space is $\kappa$-compact if every maximal zero-filter, that is $\kappa$-complete, is fixed; $[0, 1]^{\kappa} \setminus \{1\}$-compact spaces are exactly $\kappa^+$-compact spaces.

If $\kappa$ is not measurable then $m_{\mathcal{H}(\kappa)} = m(\kappa)$

Theorem

$m(\kappa) \leq \text{coord}(\mathcal{H}(\kappa)) \leq \inf \{ \kappa ; \kappa \text{ is } \kappa\text{-strongly compact} \}$.
Dieudonné complete spaces

Dieudonné complete space is a topological space induced by a complete uniformity.

The class $\text{Dieud}$ of Dieudonné complete spaces is simple iff the class of measurable cardinals is a set (the class is generated by $\{H(m)\}$, $m$ measurable).

By $\text{Dieud}_\kappa$ we denote the class of Dieudonné complete spaces $X$ generated by $\{H(\lambda); \lambda < \kappa\}$ (i.e., every closed discrete set in $X$ is of cardinality less than $m(\kappa)$).

Theorem

1. $\text{coord}_d(\text{Dieud}) = \infty$.
2. $\text{coord}_d(\text{Dieud}_\kappa) = m(\kappa)$
3. $m(\kappa) \leq \text{coord}(\text{Dieud}_\kappa) \leq \inf\{\mu; \mu \text{ is } m(\kappa)-\text{strongly compact}\}$. 
Dieudonné complete spaces

Dieudonné complete space is a topological space induced by a complete uniformity.

The class \( \text{Dieud} \) of Dieudonné complete spaces is simple iff the class of measurable cardinals is a set (the class is generated by \( \{ H(m) \} \), \( m \) measurable).

By \( \text{Dieud}_\kappa \) we denote the class of Dieudonné complete spaces \( X \) generated by \( \{ H(\lambda); \lambda < \kappa \} \) (i.e., every closed discrete set in \( X \) is of cardinality less than \( m(\kappa) \)).

**Theorem**

1. \( \text{coord}_d(\text{Dieud}) = \infty \).
2. \( \text{coord}_d(\text{Dieud}_\kappa) = m(\kappa) \)
3. \( m(\kappa) \leq \text{coord}(\text{Dieud}_\kappa) \leq \inf\{ \mu; \mu \text{ is } m(\kappa)-\text{strongly compact} \} \).
Dieudonné complete spaces

Dieudonné complete space is a topological space induced by a complete uniformity.

The class $\text{Dieud}$ of Dieudonné complete spaces is simple iff the class of measurable cardinals is a set (the class is generated by $\{H(m)\}$, $m$ measurable).

By $\text{Dieud}_\kappa$ we denote the class of Dieudonné complete spaces $X$ generated by $\{H(\lambda); \lambda < \kappa\}$ (i.e., every closed discrete set in $X$ is of cardinality less than $m(\kappa)$).

Theorem

1. $\text{coord}_d(\text{Dieud}) = \infty$.
2. $\text{coord}_d(\text{Dieud}_\kappa) = m(\kappa)$
3. $m(\kappa) \leq \text{coord}(\text{Dieud}_\kappa) \leq \inf\{\mu; \mu \text{ is } m(\kappa)-\text{strongly compact}\}$. 
Dieudonné complete spaces

Dieudonné complete space is a topological space induced by a complete uniformity.

The class $\text{Dieud}$ of Dieudonné complete spaces is simple iff the class of measurable cardinals is a set (the class is generated by $\{H(m)\}$, $m$ measurable).

By $\text{Dieud}_\kappa$ we denote the class of Dieudonné complete spaces $X$ generated by $\{H(\lambda); \lambda < \kappa\}$ (i.e., every closed discrete set in $X$ is of cardinality less than $m(\kappa)$).

Theorem

1. $\text{coord}_d(\text{Dieud}) = \infty$.
2. $\text{coord}_d(\text{Dieud}_\kappa) = m(\kappa)$
3. $m(\kappa) \leq \text{coord}(\text{Dieud}_\kappa) \leq \inf\{\mu; \mu \text{ is } m(\kappa)\text{-strongly compact}\}$. 
Dieudonné complete spaces

Dieudonné complete space is a topological space induced by a complete uniformity.

The class \( \text{Dieud} \) of Dieudonné complete spaces is simple iff the class of measurable cardinals is a set (the class is generated by \( \{H(m)\} \), \( m \) measurable).

By \( \text{Dieud}_\kappa \) we denote the class of Dieudonné complete spaces \( X \) generated by \( \{H(\lambda); \lambda < \kappa\} \) (i.e., every closed discrete set in \( X \) is of cardinality less than \( m(\kappa) \)).

Theorem

1. \( \text{coord}_d(\text{Dieud}) = \infty \).
2. \( \text{coord}_d(\text{Dieud}_\kappa) = m(\kappa) \)
3. \( m(\kappa) \leq \text{coord}(\text{Dieud}_\kappa) \leq \inf\{\mu; \mu \text{ is } m(\kappa)\text{-strongly compact}\} \).
Dieudonné complete spaces

Dieudonné complete space is a topological space induced by a complete uniformity.

The class $\text{Dieud}$ of Dieudonné complete spaces is simple iff the class of measurable cardinals is a set (the class is generated by $\{H(m)\}$, $m$ measurable).

By $\text{Dieud}_\kappa$ we denote the class of Dieudonné complete spaces $X$ generated by $\{H(\lambda); \lambda < \kappa\}$ (i.e., every closed discrete set in $X$ is of cardinality less than $m(\kappa)$).

**Theorem**

1. $\text{coord}_d(\text{Dieud}) = \infty$.
2. $\text{coord}_d(\text{Dieud}_\kappa) = m(\kappa)$
3. $m(\kappa) \leq \text{coord}(\text{Dieud}_\kappa) \leq \inf\{\mu; \mu \text{ is } m(\kappa)\text{-strongly compact}\}$. 
Uniform spaces

Theorem (Viddossich, 1970)

Every uniformly continuous map from a subspace of a product of uniform spaces into a uniform space $A$ depends on at most $w_u(A)$ many coordinates and the factorized map is uniformly continuous.

Theorem

Let $C$ be a productive and closed hereditary subcategory of $\text{Unif}_2$ with generators $A$. Then $\text{coord}_d(C, A) \leq (\sup \{w_u(A); A \in A\})^+$. If $A$ consist of complete spaces then $\text{coord}(C, A) \leq (\sup \{w_u(A); A \in A\})^+$.

Corollary

1. For every simple subcategory $C$ of $\text{Unif}_2$ one has $\text{coord}_d(C) < \infty$.
2. If $C$ is the class of all complete uniform spaces then $\text{coord}(C) = \omega_1$. 
Uniform spaces

Theorem (Viddossich, 1970)

Every uniformly continuous map from a subspace of a product of uniform spaces into a uniform space $A$ depends on at most $w_u(A)$ many coordinates and the factorized map is uniformly continuous.

Theorem

Let $C$ be a productive and closed hereditary subcategory of $\text{Unif}_2$ with generators $A$. Then $\text{coord}_d(C, A) \leq \left(\sup\{w_u(A); A \in A\}\right)^+$. If $A$ consist of complete spaces then $\text{coord}(C, A) \leq \left(\sup\{w_u(A); A \in A\}\right)^+.$

Corollary

1. For every simple subcategory $C$ of $\text{Unif}_2$ one has $\text{coord}_d(C) < \infty$.
2. If $C$ is the class of all complete uniform spaces then $\text{coord}(C) = \omega_1$. 
Uniform spaces

Theorem (Viddossich, 1970)

Every uniformly continuous map from a subspace of a product of uniform spaces into a uniform space $A$ depends on at most $w_u(A)$ many coordinates and the factorized map is uniformly continuous.

Theorem

Let $C$ be a productive and closed hereditary subcategory of $\text{Unif}_2$ with generators $A$. Then $\text{coord}_d(C, A) \leq (\sup\{w_u(A); A \in A\})^+$. If $A$ consist of complete spaces then $\text{coord}(C, A) \leq (\sup\{w_u(A); A \in A\})^+$.

Corollary

1. For every simple subcategory $C$ of $\text{Unif}_2$ one has $\text{coord}_d(C) < \infty$.
2. If $C$ is the class of all complete uniform spaces then $\text{coord}(C) = \omega_1$. 
Consider the category $\text{Prec}$ of all precompact (totally bounded) uniform Hausdorff spaces. It has an interesting class of generators, namely $P_\kappa = [0, 1]^\kappa \setminus \{1\}$ for all infinite cardinals $\kappa$.

Theorem

$\text{coord}_d(\text{Prec}, \{P_\kappa\}) = \infty$.

Theorem

$\kappa \leq \text{coord}(\text{Prec}(P_\kappa), P_\kappa) \leq \inf\{\lambda; \lambda \text{ is a } \kappa\text{-strongly compact cardinal}\}$.
Precompact spaces

Consider the category $\text{Prec}$ of all precompact (totally bounded) uniform Hausdorff spaces. It has an interesting class of generators, namely $P_\kappa = [0, 1]^\kappa \setminus \{1\}$ for all infinite cardinals $\kappa$.

Theorem

$\text{coord}_d(\text{Prec}, \{P_\kappa\}) = \infty$.

Theorem

$\kappa \leq \text{coord}(\text{Prec}(P_\kappa), P_\kappa) \leq \inf\{\lambda; \lambda \text{ is a } \kappa\text{-strongly compact cardinal}\}$. 
Precompact spaces

Consider the category \( \text{Prec} \) of all precompact (totally bounded) uniform Hausdorff spaces. It has an interesting class of generators, namely \( P_\kappa = [0, 1]^\kappa \setminus \{1\} \) for all infinite cardinals \( \kappa \).

**Theorem**

\[
\text{coord}_d(\text{Prec}, \{P_\kappa\}) = \infty.
\]

**Theorem**

\[
\kappa \leq \text{coord}(\text{Prec}(P_\kappa), P_\kappa) \leq \inf\{\lambda; \lambda \text{ is a } \kappa\text{-strongly compact cardinal}\}.
\]
Problems

Is $C_m = \mathcal{U}_m$ for measurable cardinals $m$? (Yes, if $m$ is $m$-strongly compact.)

Are the classes $\mathcal{U}_\kappa$ simple? (Not, if $m_1$ does not exists.)

Is the class of all zerodimensional realcompact spaces simple? (Not, if $m_1$ does not exists.)

Find a topological characterization of the property $\nu(X \times Y) = \nu(X) \times \nu(Y)$.

Is there a nontrivial productive class in $\text{Top}$ closed under quotients and disjoint sums? (Not, if $s$ does not exists.)
Is \( C_m = U_m \) for measurable cardinals \( m \)? (Yes, if \( m \) is \( m \)-strongly compact.)

Are the classes \( U_\kappa \) simple? (Not, if \( m_1 \) does not exist.)

Is the class of all zerodimensional realcompact spaces simple? (Not, if \( m_1 \) does not exist.)

Find a topological characterization of the property \( \nu(X \times Y) = \nu(X) \times \nu(Y) \).

Is there a nontrivial productive class in Top closed under quotients and disjoint sums? (Not, if \( s \) does not exist.)
Problems

Is $C_m = U_m$ for measurable cardinals $m$? (Yes, if $m$ is $m$-strongly compact.)

Are the classes $U_\kappa$ simple? (Not, if $m_1$ does not exist.)

Is the class of all zerodimensional realcompact spaces simple? (Not, if $m_1$ does not exist.)

Find a topological characterization of the property $\nu(X \times Y) = \nu(X) \times \nu(Y)$.

Is there a nontrivial productive class in $\text{Top}$ closed under quotients and disjoint sums? (Not, if $5$ does not exist.)
Problems

Is $C_m = U_m$ for measurable cardinals $m$? (Yes, if $m$ is $m$-strongly compact.)

Are the classes $U_\kappa$ simple? (Not, if $m_1$ does not exist.)

Is the class of all zerodimensional realcompact spaces simple? (Not, if $m_1$ does not exist.)

Find a topological characterization of the property $\nu(X \times Y) = \nu(X) \times \nu(Y)$.

Is there a nontrivial productive class in $\text{Top}$ closed under quotients and disjoint sums? (Not, if $s$ does not exist.)
Problems

Is $C_m = U_m$ for measurable cardinals $m$? (Yes, if $m$ is $m$-strongly compact.)

Are the classes $U_\kappa$ simple? (Not, if $m_1$ does not exists.)

Is the class of all zerodimensional realcompact spaces simple? (Not, if $m_1$ does not exists.)

Find a topological characterization of the property $\nu(X \times Y) = \nu(X) \times \nu(Y)$.

Is there a nontrivial productive class in Top closed under quotients and disjoint sums? (Not, if $s$ does not exists.)