Theorem (S. Mazur 1946)

Every sequentially continuous linear functional $f$ on the power $\mathbb{R}^X$ is of the form $f(\varphi) = c_1\varphi(p_1) + \ldots + c_k\varphi(p_k)$ for some integer $k$, points $p_1, \ldots, p_k$ of $X$ and reals $c_1, \ldots, c_k$, iff $|X|$ is Ulam non-measurable.

It means that $f$ depends on finitely many coordinates. The space $\mathbb{R}^X$ may be regarded as $C_p(X)$ with a discrete space $X$. The fact that $|X|$ is Ulam non-measurable is equivalent to realcompactness of $X$.

A topological linear space is called a Mazur space if every its sequentially continuous functional is continuous.

Theorem (V. Pták, S. Mrówka, 1956)

The space $C_p(X)$ is a Mazur space iff $X$ is realcompact.
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The space $C_p(X)$ is a Mazur space iff $X$ is realcompact.
Sequential continuity on products of TLS

Question

When a product of Mazur spaces is a Mazur space?

Theorem (S. Dierolf 1975)

In a coreflective subcategory $C$ of locally convex spaces (or topological linear spaces) containing $\mathbb{R}$, a product of $\kappa$ of its nontrivial members belong to $C$ iff $\mathbb{R}^\kappa \in C$. 
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Corollary
A product of Mazur spaces is Mazur iff the number of nonzero coordinate spaces is Ulam non-measurable.
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Corollary
A product of Mazur spaces is Mazur iff the number of nonzero coordinate spaces is Ulam non-measurable.

Theorem
Productivity number of the coreflective class of Mazur spaces in TLS (or LCS) is $m_1$. 
Theorem (MH 2004)

Productivity numbers of coreflective classes in LCS are precisely measurable cardinals and $\infty$. For every measurable $m$ there is a coreflective class in LCS with its productivity number equal to $m$.

Corollary

If a coreflective class in LCS is countably productive, it is $m_1$-productive.
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Theorem (S. Mazur 1952)

Let $X_i, i \in I$ be metrizable separable spaces, $Y$ be a space having $G_\delta$ diagonal. Then every sequentially continuous map $f : \prod_I X_i \to Y$ is continuous provided $|I|$ is smaller than the first uncountable inaccessible cardinal.
Sequential continuity on products in Top

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Mazur proves that \( f \) depends on countably many coordinates, say on a countable \( J \subset I \). Then \( f_J \) is sequentially continuous, thus continuous (as defined on metrizable space), thus the composition equal to \( f \) is continuous.
Sequential continuity on products in Top

Theorem (N. Noble 1970)

A mapping on product $\prod_{i} X_i$ of topological spaces into a regular space $Y$ is continuous iff its restrictions to all $\Sigma$-products and to all canonical $2^{|I|}$ are continuous.

A sequentially continuous mapping on product $\prod_{i} X_i$ of first countable spaces is continuous iff it is continuous on all canonical subspaces $2^{|I|}$.
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Sequential cardinal

Definition (N.Noble 1970)

The first cardinal $\kappa$ such that there exists a non-continuous sequentially continuous map $2^\kappa \to \mathbb{R}$ is called sequential and denoted as $\mathfrak{s}$.

Theorem (N.Noble 1970)

Every sequentially continuous mapping on a product $\prod_i X_i$ of first-countable spaces is continuous provided $|I| < \mathfrak{s}$.

Theorem (S.Mazur 1952)

The cardinal $\mathfrak{s}$ is inaccessible.

Theorem (D.V.Chudnovskij 1977)

If we denote by $\{\kappa_\alpha\}$ the increasing system of all inaccessible numbers then $\mathfrak{s}$ is bigger than any $\kappa_\alpha$ for $\alpha < \mathfrak{s}$. Either $\mathfrak{s} \leq 2^\omega$ or $\mathfrak{s} = m_1$. 
Sequential cardinal

**Definition (N.Noble 1970)**

The first cardinal \( \kappa \) such that there exists a non-continuous sequentially continuous map \( 2^\kappa \rightarrow \mathbb{R} \) is called **sequential** and denoted as \( s \).

Since a real measure on a set is non-continuous and sequentially continuous, \( s \leq m_\mathbb{R} \).

Keisler-Tarski problem: Is \( s = m_\mathbb{R} \)?

**Theorem (N.Noble 1970)**

Every sequentially continuous mapping on a product \( \prod_i X_i \) of first-countable spaces is continuous provided \( |I| < s \).

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**Theorem (D.V.Chudnovskij 1977)**

If \( \kappa \) is a regular, then the sum is a union of all inaccessible \( \kappa \) such that \( s \neq \kappa \).
**Sequential cardinal**

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Theorem (N. Varopoulos 1964)

**Every sequentially continuous homomorphism between compact groups of cardinalities less than $m_1$ is continuous.**

Theorem

Every sequentially continuous mapping between compacts groups of cardinalities less than $m_1$ is continuous iff $s = m_1$ (the equality is valid, e.g., under CH or MA).

Theorem

Every sequentially continuous homomorphism of a product $\prod_{\lambda} G_\alpha \to G$, where $\lambda < m_1$ and $G$ is a compact group, is continuous.

It suffices to assume that $G_\alpha$ are sequential groups. The result is not true if $s < m_1$ and $G$ not compact.

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A mapping $f : X \to Y$ between uniform spaces is said to be uniformly sequentially continuous if it preserves adjacent sequences, i.e., if $\lim d(x_n, y_n) = 0$ for every uniformly continuous pseudometric $d$ on $X$ then $\lim e(f(x_n), f(y_n)) = 0$ for every uniformly continuous pseudometric $e$ on $Y$.

The first cardinal $\kappa$ such that there exists a non-continuous uniformly sequentially continuous map $2^\kappa \to \mathbb{R}$ is called uniformly sequential and denoted as $\mathfrak{s}_u$.

Every uniformly sequentially continuous mapping is sequentially continuous. Thus $\mathfrak{s} \leq \mathfrak{s}_u$.

A uniform space $X$ is said to be uniformly sequential if every uniformly sequentially continuous map on $X$ into a metric space is uniformly continuous.
Uniform sequential cardinal

**Definition**
A mapping $f : X \rightarrow Y$ between uniform spaces is said to be **uniformly sequentially continuous** if it preserves adjacent sequences, i.e., if $\lim d(x_n, y_n) = 0$ for every uniformly continuous pseudometric $d$ on $X$ then $\lim e(f(x_n), f(y_n)) = 0$ for every uniformly continuous pseudometric $e$ on $Y$.

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Theorem (MH, M.D.Rice 1978)

*Every product of less than $s_u$ of uniformly sequential spaces is uniformly sequential.*
Group sequential cardinal

**Definition**

The first cardinal $\kappa$ such that there exists a non-continuous sequentially continuous homomorphism on $\mathbb{Z}_2^\kappa$ (or on $\mathbb{Z}^\kappa$) is called group sequential and denoted as $s_g$.

Instead of non-continuous sequentially continuous homomorphism one may take non-continuous sequentially continuous pseudonorm.

Every sequentially continuous homomorphism is uniformly sequentially continuous. Thus $s_u \leq s_g$ and also $s_g \leq m_R$ (since the measure on $m_R$ witnessing the definition of $m_R$ is a non-continuous sequentially continuous pseudonorm $\mathbb{Z}_2^{m_R} \to \mathbb{R}$. Thus $s \leq s_u \leq s_g \leq m_R$.

**Theorem (B.Balcar 1995)**

There is a nontrivial Maharam submeasure on $s$, i.e., there is a nontrivial increasing, non-negative, subadditive and sequentially continuous map.
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Every sequentially continuous homomorphism is uniformly sequentially continuous. Thus $s_u \leq s_g$ and also $s_g \leq m_\mathbb{R}$ (since the measure on $m_\mathbb{R}$ witnessing the definition of $m_\mathbb{R}$ is a non-continuous sequentially continuous pseudonorm $\mathbb{Z}_2^{m_\mathbb{R}} \to \mathbb{R}$). Thus

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\[ s \leq s_u \leq s_g \leq m_R. \]

**Theorem (B. Balcar 1995)**

There is a nontrivial Maharam submeasure on \( s \), i.e., there is a nontrivial increasing, non-negative, subadditive and sequentially continuous map \( \mu : 2^s \to \mathbb{R} \) with \( \mu(\emptyset) = 0 \).

**Corollary**

The cardinalities \( s, s_u, s_g \) coincide.
Every sequentially continuous homomorphism is uniformly sequentially continuous. Thus $s_u \leq s_g$ and also $s_g \leq m_R$ (since the measure on $m_R$ witnessing the definition of $m_R$ is a non-continuous sequentially continuous pseudonorm $\mathbb{Z}^m_R \to \mathbb{R}$. Thus

$$s \leq s_u \leq s_g \leq m_R.$$
Submeasures

Definition (Submeasure)

A submeasure is a mapping \( \mu : \exp X \to [0, \infty] \) having the next properties:

1. \( \mu(\emptyset) = 0; \)
2. if \( A \subset B \) then \( \mu(A) \leq \mu(B); \)
3. \( \mu(A \cup B) \leq \mu(A) + \mu(B). \)

For an infinite \( \kappa \), \( \mu \) is said to be \( \kappa \)-subadditive if \( \mu(\bigcup_{\lambda} A_\alpha) \leq \sum_{\lambda} \mu(A_\alpha) \) whenever \( \lambda < \kappa \).

Definition (Submeasurable cardinal)

A cardinal \( \kappa \) is called submeasurable if there exists a non-zero \( \kappa \)-continuous submeasure on \( \kappa \) having zero values at singletons.

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A submeasurable cardinal is either not bigger than \( 2^\omega \) or is measurable.
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The smallest submeasurable cardinal is \( \omega \), the next one is the sequential cardinal \( s. \)

**Theorem**

A submeasurable cardinal is either not bigger than \( 2^\omega \) or is measurable.
Submeasures

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Productivity numbers in Top

**Theorem (MH 2003)**

Let $\mathcal{K}$ be a nontrivial epireflective class in the category $\text{Top}$ of topological spaces. A finitely productive coreflective class $\mathcal{C}$ of $\mathcal{K}$ is $\kappa$-productive iff $2^\lambda \in \mathcal{C}$ for all $\lambda < \kappa$.

The class of productivity numbers of coreflective classes in $\mathcal{K}$ coincides with the class of submesurable cardinals and $\{2, \infty\}$.

For every submesurable cardinal $\kappa$ there exists a coreflective class in $\text{Top}$ having $\kappa$ for its productivity number.

**Problem (MH 1988):** Is there a nontrivial productive class in $\text{Top}$ closed under quotients and disjoint sums?

**Theorem (A.Dow, S.Watson 1993)**

If GCH holds and there are no inaccessible cardinals, then every productive coreflective subcategory of $\text{Top}$ coincides with $\text{Top}$.

Existence of a proper productive coreflective subcategory of $\text{Top}$ implies existence of some large cardinal.

**Theorem (MH 2007)**

If there is no sequential cardinal then the only productive coreflective class in $\text{Top}$ is $\text{Top}$ itself.

Then every topological space is generated from a converging sequence by using finite products, disjoint sums and quotients.
Productivity numbers in Top

Problem (MH 1988): *Is there a nontrivial productive class in Top closed under quotients and disjoint sums?*

Theorem (A.Dow, S.Watson 1993)
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Theorem (H.Herrlich, MH 1999)

Let $\mathcal{K}$ be a surreflective subcategory of the category of topological groups containing $\mathbb{Z}$. A bicoreflective class $\mathcal{C}$ in $\mathcal{K}$ is $\kappa$-productive iff $\mathbb{Z}^\lambda \in \mathcal{C}$ for all $\lambda < \kappa$.

Productivity numbers of bicoreflective classes in $\mathcal{K}$ are submeasurable cardinals and $\infty$ and all are attained.

Let groups from $\mathcal{K}$ have the property that for every sequence $\{x_n\}$ of nonzero elements there exists an infinite set $S \subset \mathbb{N}$ and a sequence $\{k_n\}$ of integers such that $\sum_S k_n x_n$ does not converge. Then a bicoreflective class in $\mathcal{K}$ is either productive or is exactly $\kappa$-productive for some measurable cardinal $\kappa$. 

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Then a bicoreflective class in $\mathcal{K}$ is either productive or is exactly $\kappa$-productive for some measurable cardinal $\kappa$.

For every infinite regular cardinal $\kappa$ there exists a monocoreflective subcategory of topological Abelian groups that is exactly $\kappa$-productive.
Theorem (MH 1997)

Let $\mathcal{K}$ be a surreflective subcategory of the category of TLS containing $\mathbb{R}$. Productivity numbers of coreflective classes in $\mathcal{K}$ are submeasurable cardinals and $\infty$ and all are attained.

Coreflective classes in LCS are either productive or their productivity number is a measurable cardinal.
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Tightness of products of fans

Definition (Tightness)

For a space $X$, $A \subset X$ and $x \in \overline{A}$ define
$\ t(x, A) = \min \{|B|; \ B \subset A, x \in \overline{B}\}, \ t(X) = \sup \{t(x, A); A \subset X, x \in \overline{A}\}.$

Definition (Fans)

Fan $S(\{\lambda_{\alpha}\}_{\lambda})$ is a quotient of a disjoint sum $\bigcup_{\lambda}(\lambda_{\alpha} \oplus 1)$, where the accumulation points of $\lambda_{\alpha} \oplus 1$ are sewed together.
If $\lambda \leq \kappa$ and all $\lambda_{\alpha} < \kappa$ we shall call $S(\{\lambda_{\alpha}\}_{\lambda})$ a $\kappa$-fan.

Theorem (MH)

In products of finitely many of $m$-fans one always has $t(x, A) < m$.
Product of $m$-many of $m$-fans has tightness equal to $m$. 
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In products of finitely many of $m$-fans one always has $t(x, A) < m$.

Product of $m$-many of $m$-fans has tightness equal to $m$. 

Definition (Tightness)

For a space $X$, $A \subset X$ and $x \in \overline{A}$ define

$$t(x, A) = \min\{|B|; B \subset A, x \in \overline{B}\}, t(X) = \sup\{t(x, A); A \subset X, x \in \overline{A}\}.$$  

Definition (Fans)

Fan $S(\{\lambda_\alpha\}_\lambda)$ is a quotient of a disjoint sum $\bigcup_\lambda (\lambda_\alpha \oplus 1)$, where the accumulation points of $\lambda_\alpha \oplus 1$ are sewed together.

If $\lambda \leq \kappa$ and all $\lambda_\alpha < \kappa$ we shall call $S(\{\lambda_\alpha\}_\lambda)$ a $\kappa$-fan.

Theorem (MH)

In products of finitely many of $m$-fans one always has $t(x, A) < m$.

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Theorem (MH 2007)

\[ \chi(\xi) \geq \sup\{ \kappa \leq 2^\omega; \kappa \text{ submeasurable} \} \text{ for every } \xi \in \beta(\mathbb{N}) \setminus \mathbb{N}. \]

Corollary

If a nontrivial net in \( \mathbb{N} \) converges in \( \beta\mathbb{N} \) then its length must be at least \( \sup\{ \kappa \leq 2^\omega; \kappa \text{ submeasurable} \} \).

Balcar 1978, Shelah 1978: \( \beta\mathbb{N} \setminus \mathbb{N} \) always contains a nontrivial converging net of length \( \omega_1 \).
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Theorem

Let $\kappa$ be an uncountable submeasurable cardinal. Then

1. Every monotone $\kappa$-family is extendible to $(\kappa + 1)$-family.
2. $\Box(\kappa)$ does not hold.
3. $E(\kappa)$ does not hold.
4. Cardinal $b$ is not submeasurable.
5. There are no $(\kappa, \lambda)$-good sets for $\lambda \leq \kappa$ in the sense of Brendle and LaBerge.