LARGE CARDINALS IN GENERAL TOPOLOGY II

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Sequential continuity on products of LCS, TLS

Theorem (S.Mazur 1946)

Every sequentially continuous linear functional f on the power $\mathbb{R}^X$ is of the form $f(\varphi) = c_1\varphi(p_1) + ... + c_k\varphi(p_k)$ for some integer k, points $p_1, ..., p_k$ of X and reals $c_1, ..., c_k$, iff $|X|$ is Ulam non-measurable.

It means that f depends on finitely many coordinates. The space $\mathbb{R}^X$ may be regarded as $C_p(X)$ with a discrete space X. The fact that $|X|$ is Ulam non-measurable is equivalent to realcompactness of X.

A topological linear space is called a Mazur space if every its sequentially continuous functional is continuous.

Theorem (V.Pták, S.Mrówka, 1956)

The space $C_p(X)$ is a Mazur space iff X is realcompact.

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Question

When a product of Mazur spaces is a Mazur space?

Theorem (S.Dierolf 1975)

In a coreflective subcategory $\mathcal{C}$ of locally convex spaces (or topological linear spaces) containing $\mathbb{R}$, a product of κ of its nontrivial members belong to $\mathcal{C}$ iff $\mathbb{R}^\kappa \in \mathcal{C}$.

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A product of Mazur spaces is Mazur iff the number of nonzero coordinate spaces is Ulam non-measurable.

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A product of Mazur spaces is Mazur iff the number of nonzero coordinate spaces is Ulam non-measurable.

Theorem

Productivity number of the coreflective class of Mazur spaces in TLS (or LCS) is $\mathfrak{m}_1$.

Productivity numbers of coreflective classes of LCS

Theorem (MH 2004)

Productivity numbers of coreflective classes in LCS are precisely measurable cardinals and ∞. For every measurable $\mathfrak{m}$ there is a coreflective class in LCS with its productivity number equal to $\mathfrak{m}$.

Corollary

If a coreflective class in LCS is countably productive, it is $\mathfrak{m}_1$-productive.

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Sequential continuity on products in Top

Theorem (S.Mazur 1952)

Let $X_i, i \in I$ be metrizable separable spaces, Y be a space having G_δ diagonal. Then every sequentially continuous map $f : \prod_I X_i \to Y$ is continuous provided $|I|$ is smaller than the first uncountable inaccessible cardinal.

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Mazur proves that f depends on countably many coordinates, say on a countable $J \subset I$. Then f_J is sequentially continuous, thus continuous (as defined on metrizable space), thus the composition equal to f is continuous.

Sequential continuity on products in Top

Theorem (N.Noble 1970)

A mapping on product $\prod_I X_i$ of topological spaces into a regular space Y is continuous iff its restrictions to all Σ-products and to all canonical $2^{|I|}$ are continuous.

A sequentially continuous mapping on product $\prod_I X_i$ of first countable spaces is continuous iff it is continuous on all canonical subspaces $2^{|I|}$.

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Sequential cardinal

Definition (N.Noble 1970)

The first cardinal κ such that there exists a non-continuous sequentially continuous map $2^\kappa \to \mathbb{R}$ is called sequential and denoted as $\mathfrak{s}$.

Theorem (N.Noble 1970)

Every sequentially continuous mapping on a product $\prod_I X_i$ of first-countable spaces is continuous provided $|I| < \mathfrak{s}$.

Theorem (S.Mazur 1952)

The cardinal $\mathfrak{s}$ is inaccessible.

Theorem (D.V.Chudnovskij 1977)

If we denote by $\{\kappa_\alpha\}$ the increasing system of all inaccessible numbers then $\mathfrak{s}$ is bigger than any κ_α for $\alpha < \mathfrak{s}$. Either $\mathfrak{s} \leq 2^\omega$ or $\mathfrak{s} = \mathfrak{m}_1$.

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Since a real measure on a set is non-continuous and sequentially continuous, $\mathfrak{s} \leq \mathfrak{m}_{\mathbb{R}}$.

Keisler-Tarski problem: Is $\mathfrak{s} = \mathfrak{m}_{\mathbb{R}}$?

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Varopoulos theorem

Theorem (N.Varopoulos 1964)

Every sequentially continuous homomorphism between compact groups of cardinalities less than $\mathfrak{m}_1$ is continuous.

Theorem

Every sequentially continuous mapping between compacts groups of cardinalities less than $\mathfrak{m}_1$ is continuous iff $\mathfrak{s} = \mathfrak{m}_1$ (the equality is valid, e.g., under CH or MA).

Theorem

Every sequentially continuous homomorphism of a product $\prod_\lambda G_\alpha \to G$, where $\lambda < \mathfrak{m}_1$ and G is a compact group, is continuous.
It suffices to assume that G_α are sequential groups. The result is not true if $\mathfrak{s} < \mathfrak{m}_1$ and G not compact.

A.V.Arkhangelskij, W.Just, G.Plebanek: Sequential continuity on dyadic compacta and topological groups (Comment.Math.Univ.Carolinae 1996)

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Uniform sequential cardinal

Definition

A mapping $f : X \to Y$ between uniform spaces is said to be uniformly sequentially continuous if it preserves adjacent sequences, i.e., if $\lim d(x_n, y_n) = 0$ for every uniformly continuous pseudometric d on X then $\lim e(f(x_n), f(y_n)) = 0$ for every uniformly continuous pseudometric e on Y.

Definition (MH, M.D.Rice 1978)

The first cardinal κ such that there exists a non-continuous uniformly sequentially continuous map $2^\kappa \to \mathbb{R}$ is called uniformly sequential and denoted as $\mathfrak{s}_u$.

Every uniformly sequentially continuous mapping is sequentially continuous. Thus $\mathfrak{s} \leq \mathfrak{s}_u$.

A uniform space X is said to be uniformly sequential if every uniformly sequentially continuous map on X into a metric space is uniformly continuous.

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Theorem (MH, M.D.Rice 1978)

Every product of less than $\mathfrak{s}_u$ of uniformly sequential spaces is uniformly sequential.

Group sequential cardinal

Definition

The first cardinal κ such that there exists a non-continuous sequentially continuous homomorphism on $\mathbb{Z}_2^\kappa$ (or on $\mathbb{Z}^\kappa$) is called group sequential and denoted as $\mathfrak{s}_g$.

Instead of non-continuous sequentially continuous homomorphism one may take non-continuous sequentially continuous pseudonorm.

Every sequentially continuous homomorphism is uniformly sequentially continuous. Thus $\mathfrak{s}_u \leq \mathfrak{s}_g$ and also $\mathfrak{s}_g \leq \mathfrak{m}_\mathbb{R}$ (since the measure on $\mathfrak{m}_\mathbb{R}$ witnessing the definition of $\mathfrak{m}_\mathbb{R}$ is a non-continuous sequentially continuous pseudonorm $\mathbb{Z}_2^{\mathfrak{m}_\mathbb{R}} \to \mathbb{R}$. Thus

$$\mathfrak{s} \leq \mathfrak{s}_u \leq \mathfrak{s}_g \leq \mathfrak{m}_\mathbb{R} .$$

Theorem (B.Balcar 1995)

There is a nontrivial Maharam submeasure on $\mathfrak{s}$, i.e., there is a nontrivial increasing, non-negative, subadditive and sequentially continuous map

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There is a nontrivial Maharam submeasure on $\mathfrak{s}$, *i.e., there is a nontrivial increasing, non-negative, subadditive and sequentially continuous map* $\mu : 2^{\mathfrak{s}} \to \mathbb{R}$ *with* $\mu(\emptyset) = 0$.

Corollary

The cardinalities $\mathfrak{s}, \mathfrak{s}_u, \mathfrak{s}_g$ *coincide.*

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Submeasures

Definition (Submeasure)

A submeasure is a mapping $\mu : \exp X \to [0,\infty]$ having the next properties:

1. $\mu(\emptyset) = 0$;
2. if $A \subset B$ then $\mu(a) \le \mu(B)$;
3. $\mu(A \cup B) \le \mu(A) + \mu(B)$.

For an infinite κ, μ is said to be κ-subadditive if $\mu(\bigcup_\lambda A_\alpha) \le \sum_\lambda \mu(A_\alpha)$ whenever $\lambda < \kappa$.

Definition (Submeasurable cardinal)

A cardinal κ is called submeasurable if there exists a non-zero κ-continuous submeasure on κ having zero values at singletons.

Theorem

A submeasurable cardinal is either not bigger than 2^ω or is measurable.

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The smallest submeasurable cardinal is ω, the next one is the sequential cardinal $\mathfrak{s}$.

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Productivity numbers in Top

Theorem (MH 2003)

Let $\mathcal{K}$ be a nontrivial epireflective class in the category Top *of topological spaces. A finitely productive coreflective class $\mathcal{C}$ of $\mathcal{K}$ is κ-productive iff $2^\lambda \in \mathcal{C}$ for all $\lambda < \kappa$.*
The class of productivity numbers of coreflective classes in $\mathcal{K}$ coincides with the class of submesurable cardinals and $\{2, \infty\}$.
*For every submeasurable cardinal $\mathfrak{t}$ there exists a coreflective class in*Top *havuing $\mathfrak{t}$ for its productivity number.*

Problem (MH 1988): *Is there a nontrivial productive class in Top closed under quotients and disjoint sums?*

Theorem (A.Dow, S.Watson 1993)

If GCH holds and there are no inaccessible cardinals, then every productive coreflective subcategory of Top *coincides with* Top.
Existence of a proper productive coreflective subcategory of Top *implies existence of some large cardinal.*

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Theorem (MH 2007)

If there is no sequential cardinal then the only productive coreflective class in Top is Top itself.

Then every topological space is generated from a converging sequence by using finite products, disjoint sums and quotients.

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Productivity numbers in TopGr

Theorem (H.Herrlich, MH 1999)

Let $\mathcal{K}$ be a surreflective subcategory of the category of topological groups containing $\mathbb{Z}$. A bicoreflective class $\mathcal{C}$ in $\mathcal{K}$ is κ-productive iff $\mathbb{Z}^\lambda \in \mathcal{C}$ for all $\lambda < \kappa$.

Productivity numbers of bicoreflective classes in $\mathcal{K}$ are submeasurable cardinals and ∞ and all are attained.

Let groups from $\mathcal{K}$ have the property that for every sequence $\{x_n\}$ of nonzero elements there exists an infinite set $S \subset \mathbb{N}$ and a sequence $\{k_n\}$ of integers such that $\sum_S k_n x_n$ does not converge.
Then a bicoreflective class in $\mathcal{K}$ is either productive or is exactly κ-productive for some measurable cardinal κ.

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Then a bicoreflective class in $\mathcal{K}$ is either productive or is exactly κ-productive for some measurable cardinal κ.

For every infinite regular cardinal κ there exists a monocoreflective subcategory of topological Abelian groups that is exactly κ-productive.

Productivity numbers in TLS

Theorem (MH 1997)

Let $\mathcal{K}$ be a surreflective subcategory of the category of TLS containing $\mathbb{R}$. Productivity numbers of coreflective classes in $\mathcal{K}$ are submeasurable cardinals and ∞ and all are attained.

Coreflective classes in LCS are either productive or their productivity number is a measurable cardinal.

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Coreflective classes in LCS are either productive or their productivity number is a measurable cardinal.

Tightness of products of fans

Definition (Tightness)

For a space $X, A \subset X$ and $x \in \overline{A}$ define
$t(x, A) = \min\{|B|; B \subset A, x \in \overline{B}\}, t(X) = \sup\{t(x, A); A \subset X, x \in \overline{A}\}$.

Definition (Fans)

Fan $S(\{\lambda_\alpha\}_\lambda)$ is a quotient of a disjoint sum $\bigcup_\lambda(\lambda_\alpha \bigoplus 1)$, where the accumulation points of $\lambda_\alpha \bigoplus 1$ are sewed together.
If $\lambda \leq \kappa$ and all $\lambda_\alpha < \kappa$ we shall call $S(\{\lambda_\alpha\}_\lambda)$ a κ-fan.

Theorem (MH)

In products of finitely many of $\mathfrak{m}$-fans one always has $t(x, A) < \mathfrak{m}$.
Product of $\mathfrak{m}$-many of $\mathfrak{m}$-fans has tightness equal to $\mathfrak{m}$.

Tightness of products of fans

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$\chi(\xi) \geq \sup\{\kappa \leq 2^\omega; \kappa$ *submeasurable* $\}$ *for every* $\xi \in \beta(\mathbb{N}) \setminus \mathbb{N}$.

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If a nontrivial net in $\mathbb{N}$ *converges in* $\beta\mathbb{N}$ *then its length must be at least* $\sup\{\kappa \leq 2^\omega; \kappa$ *submeasurable* $\}$.

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Theorem

Let κ be an uncountable submeasurable cardinal. Then

1. *Every monotone κ-family is extendible to $(\kappa+1)$-family.*
2. *$\square(\kappa)$ does not hold.*
3. *$E(\kappa)$ does not hold.*
4. *Cardinal $\mathfrak{b}$ is not submeasurable.*
5. *There are no (κ, λ)-good sets for $\lambda \leq \kappa$ in the sense of Brendle and LaBerge.*

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