LARGE CARDINALS IN GENERAL TOPOLOGY I

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The existence of a large cardinal $\kappa$ should not be inconsistent with ZFC. If ZFC is consistent, then ZFC + "the large cardinal $\kappa$ does not exist" is consistent.
Measurable cardinals
S.Banach, K.Kuratowski, S.Ulam (Lvov 1929-1930)

Stanislaw Marcin Ulam, 1909-1984
Stefan Banach 1892–1945
Kazimierz Kuratowski 1896–1980

Definition

For an infinite cardinal $\kappa$ we say that a measure $\mu$ on $A$ is $\kappa$-additive if
\[ \mu(\bigcup \lambda A_\alpha) = \sum \lambda \mu(A_\alpha) \]
whenever $\{A_\alpha\}_\lambda$ is a disjoint collection of subsets of $A$ and $\lambda < \kappa$.

If $\kappa = \omega$ (or $\kappa = \omega_1$), we speak about finitely additive (or countably additive) measure.

Definition (Measurable cardinals)

A cardinal number $\kappa$ is said to be real-measurable if there is a $\kappa$-additive measure on the set $\kappa$.

A cardinal number $\kappa$ is said to be measurable if there is a $\kappa$-additive two-valued measure on the set $\kappa$.

The class of measurable cardinals will be ordered:
$\omega = m_0 < m_1 < ...$.

The first uncountable real-measurable cardinal is denoted as $m_R$. 
All our measures are defined on all subsets of some set. We shall assume that measures are non-trivial in the sense that the measure of the whole set is not zero, while measures of points are zero.

Measures with ranges equal to \{0, 1\} are called two-valued.
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The class of measurable cardinals will be ordered: \( \omega = m_0 < m_1 < \ldots \). The first uncountable real-measurable cardinal is denoted as \( m_\mathbb{R} \).
**Theorem**

*Every real measurable cardinal is inaccessible.*

*Every measurable cardinal is strongly inaccessible.*

*Any real measurable cardinal is measurable provided it is bigger than $2^\omega$."

**Theorem (R.N.Solovay)**

*The consistencies of $\{\text{ZFC } + \exists m_1\}$, $\{\text{ZFC } + \exists m_R\}$, $\{\text{ZFC } + (\exists m_R \leq 2^\omega)\}$ are equivalent.*

For a cardinal $\kappa$, we denote by $m(\kappa)$ the first measurable cardinal bigger than $\kappa$ or a symbol $\infty$ bigger than any cardinal if there is no measurable cardinal bigger than $\kappa$.

**Theorem**

*Every $\kappa$-additive measure is $m(\kappa)$-additive.*

**Corollary**

*The first uncountable measurable cardinal is the first uncountable cardinal admitting a countably additive measure.*
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Ultrafilters

**Definition (κ-completeness of filters)**

For an infinite cardinal $\kappa$, a filter $\mathcal{F}$ of subsets of $A$ is said to be $\kappa$-complete if $\bigcap_\lambda A_\alpha \in \mathcal{F}$ whenever $\{A_\alpha\}_\lambda \subset \mathcal{F}$ and $\lambda < \kappa$.

Instead of $\omega_1$-complete filters we speak about countably complete filters.

If $\mu$ is a two-valued $\kappa$-additive measure on a set $A$ then $\{P \subset A; \mu(P) = 1\}$ is a free $\kappa$-complete ultrafilter on $A$.

Conversely, if $\mathcal{F}$ is a free $\kappa$-complete ultrafilter then $\mu$ with value 1 at sets from $\mathcal{F}$ and zero otherwise is a two-valued $\kappa$-additive measure on $A$.

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A cardinal $\kappa$ is measurable iff there exists a free $\kappa$-complete ultrafilter on the set $\kappa$. 
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## Ultrafilters

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### Theorem

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Measurable cardinals and topology

In this part, we shall work in Hausdorff completely regular (i.e., Tikhonov) spaces only.
Trivially: A set $X$ has cardinality less than $m$ iff every maximal filter on $\exp X$ is fixed provided it is $m$-complete.

When $X$ is a topological space, one can take some subclasses of $\exp X$ instead of $\exp X$, like the class of closed sets or of zero sets in $X$ (i.e., sets of a form $f^{-1}(0)$, $f : X \to \mathbb{R}$ continuous).
Theorem \((m = \omega)\)

The following conditions for a topological space \(X\) are equivalent:

1. \(X\) is compact.
2. Every ultrafilter \(\mathcal{U}\) on \(X\) converges (i.e., \(\bigcap_{\mathcal{U}} \overline{A} \neq \emptyset\)).
3. For every ultrafilter \(\mathcal{U}\) on \(X\) the filter \(\{F \in \mathcal{U}; F\) is closed\} in closed sets in \(X\) is fixed (has nonempty intersection).
4. Every maximal filter of closed sets in \(X\) is fixed (has nonempty intersection).
5. For every ultrafilter \(\mathcal{U}\) on \(X\) the filter \(\{F \in \mathcal{U}; F\) is a zero set\} in zero sets in \(X\) is fixed (has nonempty intersection).
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Zero sets

For every ultrafilter $\mathcal{X}$ on $X$ the filter $\{ F \in \mathcal{X}; F \text{ is a zero set } \}$ in zero sets in $X$ is fixed (has nonempty intersection) provided it is $\kappa$-complete.

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Zero sets

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Definition ($\kappa$-compact spaces, H.Herrlich)

A topological space $X$ is said to be $\kappa$-compact if every maximal zero filter that is $\kappa$-complete, has nonempty intersection.

$\omega_1$-compact space $=$ realcompact space
Closed sets

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Every maximal filter of closed sets in \( X \) is fixed (has nonempty intersection) provided it is \( \kappa \)-complete.

Definition (\( \kappa \)-ultracompact spaces, J.van der Slot)

A topological space \( X \) is said to be \( \kappa \)-ultracompact if every ultrafilter with \( \kappa \)-complete property for its closed sets converges.
Definition

A productive and closed-hereditary (i.e., epireflective) class $\mathcal{C}$ of spaces is said to be **simple** if there is $Z \in \mathcal{C}$ such that every $X \in \mathcal{C}$ can be embedded onto a closed subspace a power of $Z$. One says that $\mathcal{C}$ is generated by $Z$ and the spaces from $\mathcal{C}$ are then called $Z$-compact spaces.

By $\mathcal{C}_\kappa$ we denote the class of $\kappa$-compact spaces.

By $\mathcal{U}_\kappa$ we denote the class of $\kappa$-ultracompact spaces.

The classes $\mathcal{C}_\omega, \mathcal{U}_\omega$ are simple, they coincide with the class of compact spaces and are generated by $[0, 1]$.

The class $\mathcal{C}_{\omega_1}$ is the simple class of realcompact spaces generated by $\mathbb{R}$.

1. Are the classes $\mathcal{C}_\kappa, \kappa \geq \omega_2$ simple?
2. Are the classes $\mathcal{U}_\kappa, \kappa \geq \omega_1$ simple?
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Theorem (MH)
The classes $\mathcal{C}_\kappa$ are simple. For any cardinal $\kappa$, the class $\mathcal{C}_{\kappa^+}$ is generated by $P_\kappa = [0, 1]^\kappa \setminus \{1\}$. For limit $\kappa$ the class $\mathcal{C}_\kappa$ is generated by $\prod_\kappa P_\lambda$.

Theorem (van der Slot, Z.Frolík)
The class of perfect images of spaces from $C_\kappa$ coincides with the class $\mathcal{U}_\kappa$.

Theorem (MH)
If $C$ is epireflective, closed under perfect images and contains a discrete space of cardinality $\mu$ then $C$ is not a part of $E$-compact spaces for any space $E$ of cardinality less than $m(\mu)$.

Corollary
1. The classes $\mathcal{U}_\kappa$, $\omega < \kappa < m_1$, are not generated by a space of cardinality $< m_1$.
2. The classes $C_\kappa$, $\omega < \kappa < m_1$, and $\mathcal{U}_\lambda$ are all different.
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*The classes* $\mathcal{C}_\kappa$ *are simple. For any cardinal* $\kappa$, *the class* $\mathcal{C}_{\kappa^+}$ *is generated by* $P_\kappa = [0, 1]^{\kappa} \setminus \{1\}$. *For limit* $\kappa$ *the class* $\mathcal{C}_\kappa$ *is generated by* $\prod_\kappa P_\lambda$.

Theorem (van der Slot, Z.Frolík)

*The class of perfect images of spaces from* $\mathcal{C}_\kappa$ *coincides with the class* $\mathcal{U}_\kappa$.

Theorem (MH)

*If* $\mathcal{C}$ *is epireflective, closed under perfect images and contains a discrete space of cardinality* $\mu$ *then* $\mathcal{C}$ *is not a part of* $E$-*compact spaces for any* $E$ *of cardinality less than* $m(\mu)$.

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PROBLEMS

1. Is $C_m = U_m$ for measurable cardinals $m$?

2. Are the classes $U_\kappa$ simple?

Similar situation

P. Nyikos: The class of $\mathbb{N}$-compact spaces is a proper subclass of the class of all zerodimensional realcompact spaces. It does not contain the Prabir Roy’s metric space $\Delta$ with $\text{ind} \, \Delta = 0$, $\text{Ind} \, \Delta = 1$.

Problem: Is the class of all zerodimensional realcompact spaces simple?

A. Mysior

The class of all zerodimensional realcompact spaces is not generated by any space of Ulam non-measurable cardinality.

Problem

Is the class of all zerodimensional realcompact spaces generated by a space of cardinality bigger than $m_1$?
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Dieudonné complete spaces

Definition
A space $X$ is said to be Dieudonné complete if there is a complete uniformity inducing its topology (i.e., the fine uniformity of $X$ is complete).

Theorem (MH)
If $X$ is a Dieudonné complete space and $m$ a measurable cardinal then the following properties are equivalent:

1. $X$ is $\kappa$-compact and $m(\kappa) = m$, $\kappa$ not measurable.
2. $X$ is $\lambda$-ultracompact and $m(\lambda) = m$, $\lambda$ not measurable.
3. $X$ contains no closed discrete subspace of cardinality $m$.
4. $X$ is $H(\mu)$-compact for any $\mu$ with $m(\mu) = m$.

Corollary
The class of Dieudonné spaces is simple iff the class of measurable cardinals is a set.
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Every sequentially continuous homomorphism between compact groups of cardinalities less than $m_1$ is continuous.

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Theorem (CH)

Every sequentially continuous mapping between compact groups of cardinalities less than $m_1$ is continuous.
Let $X \subset \prod_{i} X_i$ and $f : X \rightarrow Y$. We say that $f$ depends on $J \subset I$ (or on $|J|$ coordinates, or that $f$ factorizes via $\text{pr}_J(X)$) if there exists a map $f_J : \text{pr}_J(X) \rightarrow Y$ such that $f = f_J \circ \text{pr}_J$, i.e., if $f(x) = f(y)$ provided $x, y \in X, \text{pr}_J(x) = \text{pr}_J(y)$.
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\[
\begin{tikzcd}
X \arrow{r} \arrow[swap]{d}{f} & \prod_i X_i \arrow{dr}{pr_J} \arrow[swap]{dl}{f_J} \arrow{d}{pr_J(X)} & \prod_J X_i \\
Y & & 
\end{tikzcd}
\]
**Σ-products**

**Σ-product**

Let $p \in \prod_I X_i$. The subset \( \{ x \in \prod_I X_i; |i \in I; \text{pr}_i a \neq \text{pr}_i(p)| \leq \omega \} \) is called a Σ-product of \( \{ X_i \}_I \) with the basic point \( p \).

If instead of \( \leq \omega \) in the previous definition we use \( < \omega \) we get σ-products.

**Theorem (N.Noble)**

If all \( X_i, i \in I \), are first countable then every Σ-product of \( \{ X_i \}_I \) is a Fréchet space.

**Corollary**

Every sequentially continuous mapping on a product of first countable spaces is continuous on every Σ-product of \( \{ X_i \}_I \).
**Σ-products**

**Σ-product**

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If instead of \( \leq \omega \) in the previous definition we use \( < \omega \) we get **σ-products**.

**Theorem (N. Noble)**

*If all \( X_i, i \in I, \) are first countable then every Σ-product of \( \{ X_i \}_I \) is a Fréchet space.*

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**Σ-products**

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If instead of $\leq \omega$ in the previous definition we use $<\omega$ we get **σ-products**.

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*If all $X_i, i \in I$, are first countable then every Σ-product of $\{X_i\}_I$ is a Fréchet space.*

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*Every sequentially continuous mapping on a product of first countable spaces is continuous on every Σ-product of $\{X_i\}_I$.***
Productivity number

**Question**

*When a sequentially continuous map defined on a product of spaces is continuous?*

Spaces having the property that every sequentially continuous map defined on them and ranging in a given class of spaces is continuous, form a coreflective class, i.e., the class is closed under taking quotients and inductive limits (sums). So, there is a more general question how big are the so called productivity numbers of coreflective classes:

**Definition**

Productivity number of a coreflective class \( \mathcal{C} \) is the smallest cardinal \( \kappa \) such that a product \( \prod_{\kappa} X_\alpha, X_\alpha \in \mathcal{C} \), does not belong to \( \mathcal{C} \).
Question

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Spaces having the property that every sequentially continuous map defined on them and ranging in a given class of spaces is continuous, form a coreflective class, i.e., the class is closed under taking quotients and inductive limits (sums). So, there is a more general question how big are the so called productivity numbers of coreflective classes:

Definition

Productivity number of a coreflective class $C$ is the smallest cardinal $\kappa$ such that a product $\prod_\kappa X_\alpha, X_\alpha \in C$, does not belong to $C$. 
Productivity number

**Question**

*When a sequentially continuous map defined on a product of spaces is continuous?*

Spaces having the property that every sequentially continuous map defined on them and ranging in a given class of spaces is continuous, form a coreflective class, i.e., the class is closed under taking quotients and inductive limits (sums). So, there is a more general question how big are the so called productivity numbers of coreflective classes:

**Definition**

*Productivity number* of a coreflective class $\mathcal{C}$ is the smallest cardinal $\kappa$ such that a product $\prod_\kappa X_\alpha$, $X_\alpha \in \mathcal{C}$, does not belong to $\mathcal{C}$. 
Theorem (I. Glicksberg)

*For infinite spaces $X, Y$ the equality $\beta(X \times Y) = \beta(X) \times \beta(Y)$ holds iff $X \times Y$ is pseudocompact.*

Theorem

Let $X, Y$ have Ulam measurable cardinalities. If $v(X \times Y) = v(X) \times v(Y)$ then $X \times Y$ is pseudo-$m_1$-compact. The converse is not true.

Theorem

The property $v(X \times Y) = v(X) \times v(Y)$ for $X \times Y$ is not topological for infinite spaces $X, Y$.

Theorem

Let $\mathcal{K}$ be a finitely productive class of spaces containing all compact spaces and a pair $P, Q$ with $v(P \times Q) \neq v(P) \times v(Q)$. Then there are no topological properties $\mathcal{A}, \mathcal{B}$ such that for $X, Y \in \mathcal{K}$ one has $v(X \times Y) = v(X) \times v(Y)$ iff $X, Y \in \mathcal{A}, X \times Y \in \mathcal{B}$. 
Theorem (I. Glicksberg)

For infinite spaces $X, Y$ the equality $\beta(X \times Y) = \beta(X) \times \beta(Y)$ holds iff $X \times Y$ is pseudocompact.

What is the situation for Hewitt-Nachbin realcompactification $\nu$? When $\nu(X \times Y) = \nu(X) \times \nu(Y)$? There is a partial analogous assertion to Glicksberg result:

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Let $X, Y$ have Ulam measurable cardinalities. If $\nu(X \times Y) = \nu(X) \times \nu(Y)$ then $X \times Y$ is pseudo-$m_1$-compact. The converse is not true.

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Let $X, Y$ have Ulam measurable cardinalities. If $\upsilon(X \times Y) = \upsilon(X) \times \upsilon(Y)$ then $X \times Y$ is pseudo-$m_1$-compact. The converse is not true.

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Let $X, Y$ have Ulam measurable cardinalities. If $\nu(X \times Y) = \nu(X) \times \nu(Y)$ then $X \times Y$ is pseudo-$m_1$-compact.

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Let $\mathcal{K}$ be a finitely productive class of spaces containing all compact spaces and a pair $P, Q$ with $\nu(P \times Q) \neq \nu(P) \times \nu(Q)$. Then there are no topological properties $\mathcal{A}, \mathcal{B}$ such that for $X, Y \in \mathcal{K}$ one has $\nu(X \times Y) = \nu(X) \times \nu(Y)$ iff $X, Y \in \mathcal{A}$, $X \times Y \in \mathcal{B}$. 