A bit more general approach to Haar-smallnes

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joint work with Eliza Jabłońska, Taras Banakh and Szymon Głąb (still in progress)

Just a bit of history

G stands for a Polish group, not necessary abelian.

Theorem (Haar, 1933)

G is locally compact if and only if there exist a left-invariant regular nontrivial Borel measure, which is in such a case unique up to multiplying by a constant.

Definition (Christensen, 1972)

We call a set $A \subset G$ *Haar-null*, if there exists such a Borel probability measure μ on G and a Borel set $B \supset A$ that for any $g, h \in G$ we have $\mu(gBh) = 0$.

Give short justification σ-ideal and locally compact case

Definition (Darji, 2013)

We call a set $A \subset G$ *Haar-meager*, if there exists such a compact metrizable K, a continuous $f : K \to G$ and a Borel set $B \supset A$ that for any $g, h \in G$ we have $f^{-1}(gBh) \in \mathcal{M}_K$.
Equivalently fix $K = 2^\omega$.

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Banakh, Głąb, E. Jabłońska, S.

Let $\mathcal{I}$ be a semi-ideal on a compact K. We call a set $A \subset G$ Haar-$\mathcal{I}$ ($A \in \mathcal{HI}$), if there exists such a continuous $f : K \to G$ and a Borel set $B \supset A$ that for any $g, h \in G$ we have $f^{-1}(gBh) \in \mathcal{I}$. We focus on cases $K \in \{2^\omega, \omega + 1\}$. put it on the table

So, what are possible versions of the above definition?

- We may look for ideals on various K's;
- We may change the class of the hull B, getting e.g. *naive* ($B \in \mathcal{P}(X)$) and *universal* versions;
- We may look for just one-handed translations (*left* or *right*);
- We may demand witnessing function to be an injection;
- We may demand the set of witnessing functions to be comeager in $C(K, G)$ (*generic*).

On one hand, it may result in monsters like "$A \in \mathcal{NEGLHI}$"
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Problem 2 (Darji 2013), solved by Elekes & co. (2018)

Assume $A \subset G$ is Haar-meager. Is there such a compact set $K \subset G$ that for all $g, h \in G$ we have $gAh \cap K \in \mathcal{M}_K$?

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Choosing the compact K

If $\mathcal{I}$ is a semi-ideal on the metrizable K, then there exists such a semi-ideal $\mathcal{J}$ on 2^ω that $\mathcal{HI} \subset \mathcal{HJ}$. Even better if open sets are not members of $\mathcal{I}$.

Connected K's are not good for totally disconnected groups.

If $K = [0,1]^n$ and $\mathcal{I}$ is a family of null (meager) subsets of K, then each $\mathcal{HI}$ set is $\mathcal{HN}$ ($\mathcal{HM}$). If $G = \mathbb{R}^m$, then those notions coincide.

Theorem

Each null-finite set is both Haar-null and injectivily Haar-meager.

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In $G = \mathbb{R}^\omega$ there exists a closed F which is null-1 but not Haar-countable.

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Complexity of hulls

Naive versions are not so good, since under CH each group of the form $X \times X$ is a union of two Haar-countable sets. In ZFC $\mathbb{R}^2$ is a countable union of Haar-1 sets.

Theorem

In non-locally compact groups for each $\xi < \omega_1$ there exists a Haar-1 set which do not admit Σ^0_ξ Haar-null (Haar-meager) hull. From the proof one can derive a more general Theorem, also our approach unifies the proofs. This shows that it is not so good idea to limit Borel complexity of allowed hulls.

Theorem

For any Borel-on-Borel σ-ideal $\mathcal{I}$ on 2^ω we have $add(\mathcal{HI}) = \omega_1$.

Theorem

For any "Σ^1_1-on-Π^1_1" ideal $\mathcal{I}$ on K each analytic naively Haar-$\mathcal{I}$ set is contained in Borel $\mathcal{HI}$ set.

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Haar-null case

Universally measureable hulls works. Elekes and Vindnyanszky proved they give bigger family.

Haar-meager case

Not so clear - there are two possibilities for defining universal Baire measureability. In general we demand all continuous preimages to be Baire measureable, but on which spaces?

Strongly unclear in other cases.

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One sided version

$\mathcal{LHI}$ is still two-sided invariant. However...

Theorem (Solecki, 2006)

Assume that G has a free subgroup at 1. Then there exists a Borel $B \in \mathcal{LHN}$ so that $G = B \cup Bg$ for some $g \in G$.

Injective witnesses

Haar-null

$\mathcal{EHN} = \mathcal{HN}$, same for the other versions.

For Haar-finiteness and Haar-countability injectivity also does not change anything.

Haar-meager

$\mathcal{EHM} \subset \mathcal{SHM} \subset \mathcal{HM}$
If G is totally disconnected, then $\mathcal{EHM} = \mathcal{SHM}$.
If G is hull-compact, then $\mathcal{SHM} = \mathcal{HM}$. In $\mathbb{R}^\omega$ we have $\mathcal{EHM} \neq \mathcal{SHM}$; also recall the recent result of Elekes & co.

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$\mathcal{EHN} = \mathcal{HN}$, same for the other versions.

For Haar-finiteness and Haar-countability injectivity also does not change anything.

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Some properties

$K = 2^\omega$.

A. Kwela's paper

Among others, for $G = \mathbb{R}$:

- Haar-finite sets does not form an ideal;
- All families of Haar-n sets and Haar-finite differs.

Fubini property

Each family $\mathcal{I}$ of subsets of the space 2^ω induces the families

$$\mathcal{I}_i^n = \{A \subset (2^\omega)^n : \forall a \in (2^\omega)^{n \setminus \{i\}} \ \ e_a^{-1}(A) \in \mathcal{I}\}.$$

We say that $\mathcal{I}$ is *Fubini* if for some (any) $n \in \mathbb{N} \cup \{\omega\}$ there exists a continuous map $h : 2^\omega \to (2^\omega)^n$ such that for any $i \in n$ and any Borel set $B \in \mathcal{I}_i^n$ the preimage $h^{-1}(B)$ belongs to the family $\mathcal{I}$. If I is a Fubini (σ)-ideal, then $\mathcal{H}\mathcal{I}$ is also a (σ)-ideal.

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$C := \{\sum_{n\in\omega} \frac{\epsilon_n}{7^n} : \forall_{n\in\omega} \epsilon_n \in \{1,2\}\}$, $D := \{\sum_{n\in\omega} \frac{\epsilon_n}{7^n} : \forall_{n\in\omega} \epsilon_n \in \{3,5\}\}$. $dim(C) = dim(D) = ln(2)/ln(7)$. Using similar mathod we may construct elements of $\mathcal{H}1$ with Hausdorff dimension arbitrary close to 1. Mattila gave example of such a sets with Hausdorff dimension 1.

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Assume that Polish group G can be decompose to form $G = \mathbb{R} \times H$. Then there exists a homeomorph $A \subset \mathcal{I}R$ of the Cantor set for which $dim(A) = 1$ and $A \times H \in \mathcal{H}1$. In particular for each $n \in \omega$ there exists $A \subset \mathbb{R}^n$, $A \in \mathcal{H}1$ with $dim(A) = n$.

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Examples

Each countable $A \subset G$ is both left and right Haar-1. Proof.

Theorem

If G is abelian and $A \subset G$ is such that $A - A$ is meager, then $A \in \mathcal{GH}1$.

The set $\{f \in C[0,1] : f \text{ is monotone on some interval}\}$ is $\mathcal{GH}Count$ and naively Haar-1.

In both $L_p[0,1]$, $L_p(\mathbb{R})$ for $p \in [1,\infty)$ the set $\{f : \int f = 0\}$ is $\mathcal{GH}1$.

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$\{f \in C[0,1] : f \text{ is somewhere one-sided differentiable}\}$ is not Haar-countable. However it follows from the Hunt's proof (1994) that it is Haar-$\mathcal{I}$ for $\mathcal{I}$ being a σ-ideal generated by closed null subsets of 2^ω. Hence also Haar-null and Haar-meager.

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Děkuji za pozornost! Köszönöm a figyelmet!
Thank you for your attention! Dziękuję za uwagę!
Хвала на пажњи! Gracias por su atención!
Gratiam vobis ago pro animis attentis!
Χάριν ὑμῖν ἔχω τῆς ὑμῶν προσοχῆς
Ďakujem za vašu pozornosť! Дякую за увагу!
Grazie per l’attenzione! Merci de votre attention !
תודה לכם על תשומת הלב Obrigado pela atenção!
Bedankt voor uw aandacht!
Danke für Ihre Aufmerksamkeit!
Diolch am eich sylw! ध्यान देने के लिए धन्यवाद!
გმადლობთ ყურადღებისთვის!