A bit more general approach to Haar-smallness

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section Set Theory & Topology,
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joint work with Eliza Jabłońska, Taras Banakh and Szymon Głąb (still in progress)
Just a bit of history

$G$ stands for a Polish group, not necessary abelian.

**Theorem (Haar, 1933)**

$G$ is locally compact if and only if there exist a left-invariant regular nontrivial Borel measure, which is in such case unique up to multiplying by a constant.

**Definition (Christensen, 1972)**

We call a set $A \subseteq G$ **Haar-null**, if there exists such a Borel probability measure $\mu$ on $G$ and a Borel set $B \supset A$ that for any $g, h \in G$ we have $\mu(gBh) = 0$.

Give short justification $\sigma$-ideal and locally compact case

**Definition (Darji, 2013)**

We call a set $A \subseteq G$ **Haar-meager**, if there exists such a compact metrizable $K$, a continuous $f : K \to G$ and a Borel set $B \supset A$ that for any $g, h \in G$ we have $f^{-1}(gBh) \in \mathcal{M}_K$.

Equivalently fix $K = 2^\omega$.

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Give short justification.
Let $\mathcal{I}$ be a semi-ideal on a compact $K$. We call a set $A \subseteq G$ Haar-$\mathcal{I}$ ($A \in \mathcal{H}\mathcal{I}$), if there exists such a continuous $f : K \to G$ and a Borel set $B \supseteq A$ that for any $g, h \in G$ we have $f^{-1}(gBh) \in \mathcal{I}$. We focus on cases $K \in \{2^{\omega}, \omega + 1\}$. Put it on the table.

So, what are possible versions of the above definition?

- We may look for ideals on various $K$’s;
- We may change the class of the hull $B$, getting e.g. naive ($B \in \mathcal{P}(X)$) and universal versions;
- We may look for just one-handed translations (left or right);
- We may demand witnessing function to be an injection;
- We may demand the set of witnessing functions to be comeager in $C(K, G)$ (generic).

On one hand, it may result in monsters like "$A \in \mathcal{N}\mathcal{E}\mathcal{G}\mathcal{L}\mathcal{H}\mathcal{I}$". On the other hand, it gives us some scale to detect how small are some small sets.
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Uniform definition

Banakh, Głąb, E. Jabłońska, S.

Let $\mathcal{I}$ be a semi-ideal on a compact $K$. We call a set $A \subset G$ Haar-$\mathcal{I}$ ($A \in \mathcal{HI}$), if there exists such a continuous $f : K \to G$ and a Borel set $B \supset A$ that for any $g, h \in G$ we have $f^{-1}(gBh) \in \mathcal{I}$. We focus on cases $K \in \{2^{\omega}, \omega + 1\}$.

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Theorem

If $\mathcal{I}$ is a $\sigma$-ideal of null subsets of $2^{\omega}$, then family of Haar-$\mathcal{I}$ sets is equal to Christensen’s family of Haar-null sets.

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**Problem 2 (Darji 2013), solved by Elekes & co. (2018)**

Assume $A \subset G$ is Haar-meager. Is there such a compact set $K \subset G$ that for all $g, h \in G$ we have $gAh \cap K \in \mathcal{M}_K$?

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Let \( \mathcal{I} \) be a semi-ideal on a compact \( K \). We call a set \( A \subseteq G \) \( \text{Haar-} \mathcal{I} \) (\( A \in \mathcal{H} \mathcal{I} \)), if there exists such a continuous \( f : K \to G \) and a Borel set \( B \supset A \) that for any \( g, h \in G \) we have \( f^{-1}(gBh) \in \mathcal{I} \). We focus on cases \( K \in \{2^\omega, \omega + 1\} \).

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Choosing the compact $K$

If $\mathcal{I}$ is a semi-ideal on the metrizable $K$, then there exists such a semi-ideal $\mathcal{J}$ on $2^\omega$ that $\mathcal{H}\mathcal{I} \subset \mathcal{H}\mathcal{J}$. Even better if open sets are not members of $\mathcal{I}$.

Connected $K$'s are not good for totally disconnected groups.

If $K = [0, 1]^n$ and $\mathcal{I}$ is a family of null (meager) subsets of $K$, then each $\mathcal{H}\mathcal{I}$ set is $\mathcal{H}N\mathcal{I}$ ($\mathcal{H}M\mathcal{I}$). If $G = \mathbb{R}^m$, then those notions coincide.

Theorem
Each null-finite set is both Haar-null and injectively Haar-meager.

Theorem
In $G = \mathbb{R}^\omega$ there exists a closed $F$ which is null-1 but not Haar-countable.
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Complexity of hulls

Naive versions are not so good, since under CH each group of the form $X \times X$ is a union of two Haar-countable sets. In ZFC $\mathbb{R}^2$ is a countable union of Haar-1 sets.

**Theorem**

In non-locally compact groups for each $\xi < \omega_1$ there exists a Haar-1 set which do not admit $\Sigma^0_\xi$ Haar-null (Haar-meager) hull. From the proof one can derive a more general Theorem, also our approach unifies the proofs. This shows that it is not so good idea to limit Borel complexity of allowed hulls.

**Theorem**

For any Borel-on-Borel $\sigma$-ideal $\mathcal{I}$ on $2^\omega$ we have $\text{add}(\mathcal{H}\mathcal{I}) = \omega_1$.

**Theorem**

For any "$\Sigma^1_1$-on-$\Pi^1_1$" ideal $\mathcal{I}$ on $K$ each analytic naively Haar-$\mathcal{I}$ set is contained in Borel $\mathcal{H}\mathcal{I}$ set.
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Complexity of hulls-universality

Haar-null case

Universally measureable hulls works. Elekes and Vindnyanszky proved they give bigger family.

Haar-meager case

Not so clear - there are two possibilities for defining universal Baire measureability. In general we demand all continuous preimages to be Baire measureable, but on which spaces?

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\( \mathcal{LHI} \) is still two-sided invariant. However...

**Theorem (Solecki, 2006)**

Assume that \( G \) has a free subgroup at 1. Then there exists a Borel \( B \in \mathcal{LHN} \) so that \( G = B \cup Bg \) for some \( g \in G \).
Injective witnesses

Haar-null

$\mathcal{EHN} = \mathcal{HN}$, same for the other versions.

For Haar-finiteness and Haar-countability injectivity also does not change anything.

Haar-meager

$\mathcal{EHM} \subset \mathcal{SHM} \subset \mathcal{HM}$

If $G$ is totally disconnected, then $\mathcal{EHM} = \mathcal{SHM}$.
If $G$ is hull-compact, then $\mathcal{SHM} = \mathcal{HM}$. In $\mathbb{R}^\omega$ we have $\mathcal{EHM} \neq \mathcal{SHM}$; also recall the recent result of Elekes & co.
Injective witnesses

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\[ \mathcal{EN} = \mathcal{N} \], same for the other versions.

For Haar-finiteness and Haar-countability injectivity also does not change anything.

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\[ K = 2^\omega. \]

A. Kwela’s paper

Among others, for \( G = \mathbb{R} \):

- Haar-finite sets does not form an ideal;
- All families of Haar-\( n \) sets and Haar-finite differs.

Fubini property

Each family \( I \) of subsets of the space \( 2^\omega \) induces the families

\[ I^n_i = \{ A \subset (2^\omega)^n : \forall a \in (2^\omega)^n \setminus \{i\} \ e_a^{-1}(A) \in I \}. \]

We say that \( I \) is Fubini if for some (any) \( n \in \mathbb{N} \cup \{\omega\} \) there exists a continuous map \( h : 2^\omega \to (2^\omega)^n \) such that for any \( i \in n \) and any Borel set \( B \in I^n_i \) the preimage \( h^{-1}(B) \) belongs to the family \( I \). If \( I \) is a Fubini \((\sigma)\)-ideal, then \( HI \) is also a \((\sigma)\)-ideal.
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**Theorem**

Let \( f : G \to H \) be a continuous surjective homomorphism. Then preimages of Haar-\( \mathcal{I} \) sets are still Haar-\( \mathcal{I} \). Doesn’t work for images.

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Examples

For $G = \mathbb{R}$ there exists a homeomorph $C$ of the Cantor set with $C \in \mathcal{H}_1$.

$C := \{ \sum_{n \in \omega} \frac{c_n}{7^n} : \forall n \in \omega \epsilon_n \in \{1, 2\} \}$, $D := \{ \sum_{n \in \omega} \frac{c_n}{7^n} : \forall n \in \omega \epsilon_n \in \{3, 5\} \}$. $\dim(C) = \dim(D) = \frac{\ln(2)}{\ln(7)}$. Using similar method we may construct elements of $\mathcal{H}_1$ with Hausdorff dimension arbitrary close to 1. Mattila gave example of such a sets with Hausdorff dimension 1.

Observation

Assume that Polish group $G$ can be decompose to form $G = \mathbb{R} \times H$. Then there exists a homeomorph $A \subset \mathbb{R} \times H$ of the Cantor set for which $\dim(A) = 1$ and $A \times H \in \mathcal{H}_1$. In particular for each $n \in \omega$ there exists $A \subset \mathbb{R}^n$, $A \in \mathcal{H}_1$ with $\dim(A) = n$. 

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Each countable \( A \subseteq G \) is both left and right Haar-1. Proof.

**Theorem**

If \( G \) is abelian and \( A \subseteq G \) is such that \( A - A \) is meager, then \( A \in \mathcal{GH}_1 \).

The set \( \{ f \in C[0, 1] : f \text{ is monotone on some interval} \} \) is \( \mathcal{GH} \text{Count} \) and naively Haar-1.

In both \( L_p[0, 1], L_p(\mathbb{R}) \) for \( p \in [1, \infty) \) the set \( \{ f : \int f = 0 \} \) is \( \mathcal{GH}_1 \).

**Kwela, Wołoszyn**

\( \{ f \in C[0, 1] : f \text{ is somewhere one-sided differentiable} \} \) is not Haar-countable. However it follows from the Hunt’s proof (1994) that it is Haar-\( \mathcal{I} \) for \( \mathcal{I} \) being a \( \sigma \)-ideal generated by closed null subsets of \( 2^\omega \). Hence also Haar-null and Haar-meager.
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